

The quartic Fermat equation in Hilbert class fields of imaginary quadratic fields

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Abstract

It is shown that the quartic Fermat equation $x^4 + y^4 = 1$ has nontrivial integral solutions in the Hilbert class field Σ of any quadratic field $K = \mathbb{Q}(\sqrt{-d})$ whose discriminant satisfies $-d \equiv 1 \pmod{8}$. A corollary is that the quartic Fermat equation has no nontrivial solution in $K = \mathbb{Q}(\sqrt{-p})$, for $p (> 7)$ a prime congruent to $7 \pmod{8}$, but does have a nontrivial solution in the odd degree extension Σ of K . These solutions arise from explicit formulas for the points of order 4 on elliptic curves in Tate normal form. The solutions are studied in detail and the results are applied to prove several properties of the Weber singular moduli introduced by Yui and Zagier.

1 Introduction.

In the paper [1] Aigner proved that the only quadratic field in which the quartic Fermat equation $x^4 + y^4 = z^4$ has a nontrivial solution is the field $K = \mathbb{Q}(\sqrt{-7})$, and that all solutions in this field reduce to the one solution

$$\left(\frac{1 + \sqrt{-7}}{2}\right)^4 + \left(\frac{1 - \sqrt{-7}}{2}\right)^4 = 1. \quad (1.1)$$

In this paper we will show that this is actually one of an infinite number of solutions of the Fermat quartic in the Hilbert class fields of a certain family of imaginary quadratic fields; namely, those fields in which the rational prime 2 splits into two prime ideals. For any number field L , let R_L denote the ring of integers in L .

Theorem 1.1. Let $K = \mathbb{Q}(\sqrt{-d})$, where $d > 0$ and $-d \equiv 1 \pmod{8}$. Further, let $(2) = \wp_2 \wp'_2$ be the factorization of 2 into conjugate prime ideals \wp_2 and \wp'_2 in K . In

the Hilbert class field Σ of K there are generators π and ξ of the principal ideals $\wp_2 R_\Sigma = (\pi)$ and $\wp'_2 R_\Sigma = (\xi)$ for which

$$\pi^4 + \xi^4 = 1. \quad (1.2)$$

The generators π and ξ may be chosen to be conjugates over \mathbb{Q} , and for $d > 7$ we have $\pm\pi \pm \xi \neq 1$. Moreover, if $\tau = \left(\frac{\Sigma/K}{\wp_2}\right)$ is the automorphism associated to the ideal \wp_2 by the Artin map for Σ/K , then $\xi = \frac{\pi^{\tau^2} + 1}{\pi^{\tau^2} - 1}$.

Note that the solution $(x, y, z) = (\pi, \xi, 1)$ of $x^4 + y^4 = z^4$ in this theorem has the property that no non-zero power of x or y lies in \mathbb{Q} . Since $\pm\pi \pm \xi \neq 1$ for $d > 7$, these solutions do not lie on the intersection of the Fermat curve with the line $x + y = 1$, except for the solution (1.1). In fact, for $d > 7$ the point (π, ξ) does not lie on *any* rational line. Thus, these solutions are nontrivial in several senses. (See [22, p. 400] and [21].) We note that the numbers π and ξ generate Σ over \mathbb{Q} (as do π^4 and ξ^4), so their degrees are equal to $2h(-d)$, where $h(-d)$ is the class number of the ring of integers R_K of K . Thus, the degrees of the solutions in Theorem 1.1 satisfy $2h(-d) \geq 4$ for $d > 7$.

For example, in the Hilbert class field $\Sigma = \mathbb{Q}(\sqrt{-3}, \sqrt{5})$ of $K = \mathbb{Q}(\sqrt{-15})$ we have the equation

$$\left(1 - \frac{\sqrt{-3} + \sqrt{5}}{2}\right)^4 + \left(1 + \frac{\sqrt{-3} - \sqrt{5}}{2}\right)^4 = 1;$$

while in the Hilbert class field $\Sigma = \mathbb{Q}(\sqrt{-3}, \sqrt{13}, \sqrt{-50 + 14\sqrt{13}})$ of $K = \mathbb{Q}(\sqrt{-39})$ we have $\pi^4 + \xi^4 = 1$ with

$$\begin{aligned} \pi &= \frac{-3 - 3\sqrt{-3} + \sqrt{13} + \sqrt{-39}}{4} + \frac{-3 + 4\sqrt{-3} + \sqrt{-39}}{12} \sqrt{-50 + 14\sqrt{13}}, \\ \xi &= \frac{-3 + 3\sqrt{-3} + \sqrt{13} - \sqrt{-39}}{4} + \frac{3 + 4\sqrt{-3} + \sqrt{-39}}{12} \sqrt{-50 + 14\sqrt{13}}. \end{aligned}$$

There is also the solution (π, ξ) in the Hilbert class field $\Sigma = \mathbb{Q}(\pi)$ of $\mathbb{Q}(\sqrt{-23})$ in which π is a root of the irreducible polynomial

$$b_{23}(x) = x^6 + x^5 + 9x^4 - 13x^3 + 18x^2 - 16x + 8$$

and ξ is the conjugate of π given as

$$\xi = -\frac{6}{7} + \frac{15}{7}\pi - \frac{5}{4}\pi^2 + \frac{85}{56}\pi^3 + \frac{3}{14}\pi^4 + \frac{9}{56}\pi^5.$$

This last solution can also be written in the form

$$\begin{aligned}\pi &= \left(-1 - \frac{7}{23}\sqrt{-23}\right)\gamma^2 + \left(1 - \frac{\sqrt{-23}}{23}\right)\gamma + \frac{1}{2} + \frac{17}{46}\sqrt{-23}, \\ \xi &= \left(2 - \frac{2}{23}\sqrt{-23}\right)\gamma^2 + \left(-1 + \frac{3}{23}\sqrt{-23}\right)\gamma - \frac{3}{2} - \frac{5}{46}\sqrt{-23},\end{aligned}$$

where γ is a root of the polynomial $x^3 - x - 1$. This solution is especially interesting, in that (1.2) has no nontrivial solution in $K = \mathbb{Q}(\sqrt{-23})$ but does have a nontrivial solution in the cubic extension Σ of K . Theorem 1.1 provides infinitely many examples of this type, since when $d = p \equiv 7 \pmod{8}$ is a prime, the class number $h(-d) = [\Sigma : K]$ is odd.

Theorem 1.2. If $p \equiv 7 \pmod{8}$ is a prime > 7 , then $x^4 + y^4 = z^4$ has no nontrivial solution in the field $K = \mathbb{Q}(\sqrt{-p})$, but does have a nontrivial solution in the odd degree extension Σ of K , where Σ is the Hilbert class field of K .

This is reminiscent of the examples given in [3], except that there quartic diophantine equations are considered which have *no* solution in \mathbb{Q} but do have solutions in cubic extensions of \mathbb{Q} .

In the proof we consider the points of order 4 on the elliptic curve

$$E_1: Y^2 + XY + bY = X^3 + bX^2,$$

the Tate normal form for a curve with a point of order 4, where $b = 1/\alpha^4$. We find expressions for all the points in $E_1[4]$ by introducing the diophantine condition

$$Fer_4: 16\alpha^4 + 16\beta^4 = \alpha^4\beta^4. \tag{1.3}$$

We show that the non-zero coordinates of points in $E_1[4]$ are in the multiplicative group generated by $-1, 2$ and the linear fractional quantities $\beta_n = \frac{\beta + 2i^n}{2\beta}$, ($1 \leq n \leq 4$, $i = \sqrt{-1}$) (see Proposition 4.1 and [26]). The expressions we give are analogous to the nice form for the points of order 4 on the Jacobi normal form, but are more convenient for our purposes. The formulas in Proposition 4.1 are also the analogue of the formulas for the points of order 3 on the Deuring normal form given in [28], which are expressed in terms of solutions of the Fermat cubic $27X^3 + 27Y^3 = X^3Y^3$. See [26] for similar expressions for the points in $E_1[8]$.

With a judicious choice of α in terms of the Dedekind η -function, namely

$$\alpha = \zeta_8^{-1} \frac{\eta(w/4)^2}{\eta(w)^2}, \quad \zeta_8 = e^{2\pi i/8},$$

for a certain element $w = (v + \sqrt{-d})/2$ of K , the curve E_1 has complex multiplication by the maximal order R_K in the field $K = \mathbb{Q}(\sqrt{-d})$. It is known [32, p. 159] that $\alpha^4 \in \Sigma$, the Hilbert class field of K ; it follows that the X -coordinates of points of order 4 on E_1 lie in the ray class field $\Sigma_4 = \Sigma(i)$ over K of conductor $f = 4$. Using our formulas for the points of order 4 on E_1 and similar formulas for the corresponding points on a second elliptic curve E_3 (see Sections 2 and 5) we show that $\beta \in \Sigma$. This leads to a solution of (1.2) with

$$\pi = \frac{\beta}{\zeta_8^j \alpha}, \quad \xi = \frac{\beta}{2}, \quad \text{for some } j \in \{1, 3, 5, 7\}.$$

In Section 8 we normalize $\zeta_8^j \alpha, \beta$ by choosing j and the sign of β so that $\xi = \beta/2$ is conjugate to π and to $(\xi + 1)/(\xi - 1)$ over \mathbb{Q} . Then we prove algebraically that $\alpha_1 = \zeta_8^j \alpha$ and the unit

$$\gamma = \frac{\beta(\beta + 2)}{4(\beta - 2)}$$

are squares in Σ , hence that $\zeta_8^{(j-1)/2} \eta(w/4)/\eta(w) \in \Sigma$. This also implies that the elliptic curve $Y^2 = X(X^2 - 4)$ has the integral solution $P = (\beta, 2(\beta - 2)\sqrt{\gamma})$ in Σ . Whenever the discriminant $-d = -p$ is prime, a simple argument shows that the sum of the conjugates of the point P for Σ/K is a point of infinite order on $Y^2 = X(X^2 - 4)$, defined over K (see Theorem 8.11). (The curves E_1 and E_3 in the above argument are isogenous, and for the given choices of d and α , the isogeny $(E_1 \rightarrow E_3)$ represents a *Heegner point* on the modular curve $X_0(4)$.)

The same arguments show that for every odd, natural number f there is a solution (π_f, ξ_f) of (1.2) in the ring class field Ω_f of K with conductor f , and that $\Omega_f = \mathbb{Q}(\pi_f) = \mathbb{Q}(\xi_f)$. Thus, there is a whole lattice of solutions to the Fermat quartic in ring class fields of K with odd conductor. Furthermore, if $f|f'$, the numbers $\pi_{f'}/\pi_f$ and $\xi_{f'}/\xi_f$ are units in $\Omega_{f'}$. (See Section 6 and Remark 4 in Section 12.) These solutions reflect the existence of the associated Heegner points of $X_0(4)$ defined over the ring class fields Ω_f .

In Section 10 we give explicit expressions for the solution (π_f, ξ_f) in terms of the Schläfli modular functions used by Weber (see [5, p. 256] and [31], [32]). We use a result of [36] to give an explicit value of $j \in \{1, 3, 5, 7\}$ in Theorem 10.6 for which $\zeta_8^{(j-1)/2} \frac{\eta(w/4)}{\eta(w)}$ lies in the ring class field Ω_f , for $w = \frac{v + \sqrt{-d}}{2}$ with $v = 1$ or 3 and $v^2 \equiv -d \pmod{16}$. This is similar to, but not contained in, results of [17] and [31] or [32, Chs. 5-6], because the above η -quotient is evaluated at the ideal \wp_2^2 , in the notation of [17]. The fact that this ideal is not relatively prime to 2 necessitates the alternate approach taken here.

The numbers π and ξ in Theorem 1.1 turn out to be fourth roots of numbers λ for which the Legendre normal form with parameter λ has complex multiplication by R_K . Using (1.3), this observation leads to the following result.

Theorem 1.3. Let λ be a complex number for which the Legendre normal form

$$E_\lambda: \quad Y^2 = X(X-1)(X-\lambda)$$

has complex multiplication by the order R_{-f^2d} of $K = \mathbb{Q}(\sqrt{-d})$ with odd conductor f and discriminant $-f^2d \equiv 1 \pmod{8}$. Then λ lies in the ring class field Ω_f of K with conductor f . If either λ or $1/\lambda$ is an algebraic integer, then λ is a fourth power in the field Ω_f . If neither λ nor $1/\lambda$ is an algebraic integer, then $-\lambda$ is a fourth power in Ω_f . The numbers λ and $\lambda - 1$ are only divisible by prime ideal divisors of 2 in Ω_f .

This theorem is related to Theorem 1.2 of [27], which says that the values of $\bar{\lambda}$ for which the Legendre normal form $E_{\bar{\lambda}}$ is supersingular in characteristic p are fourth powers in the finite field \mathbb{F}_{p^2} . (See also Landweber [24].) Theorem 1.3 is an analogue of this result in characteristic 0. Since some square-root of the form $\sqrt{-d}$ with $d \equiv 7 \pmod{8}$ embeds into the endomorphism ring of any supersingular curve in characteristic $p > 2$, the theorem of [27] follows from Theorem 1.3 by reduction (see Theorems 7.1 and 9.1).

A secondary purpose of this paper is to relate our results to the paper [36] of Yui and Zagier, in which they give generators of small height for ring class fields Ω_f of the quadratic fields $K = \mathbb{Q}(\sqrt{-d})$, with $-d = -f^2 d_1 \equiv 1 \pmod{8}$ and $(d, 3) = 1$. Yui and Zagier define the Weber singular modulus $f_w(Q)$, for any primitive quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ of discriminant $-d$, as one of the Schläfli modular functions $\mathfrak{f}, \mathfrak{f}_1, \mathfrak{f}_2$ (depending on the parity of a and c) evaluated at the root of $Q(x, 1) = 0$ which lies in the upper half-plane, times a well-chosen 48-th root of unity. They show that $f_w(Q) = f_w(\mathcal{A})$ only depends on the $SL_2(\mathbb{Z})$ equivalence class of the form Q , or in other words, only on the ideal class \mathcal{A} in the order R_{-d} to which the form Q corresponds. In Section 10 we prove their conjecture [36, p. 1648] that the numbers $f_w(\mathcal{A})$ are conjugate over \mathbb{Q} , as \mathcal{A} ranges over the ideal classes in R_{-d} , and use this fact to prove the following.

Theorem 1.4. Let \mathcal{T} be the ideal class of R_{-d} containing the ideal $\wp_2 \cap R_{-d} = 2\mathbb{Z} + \left(\frac{v+\sqrt{-d}}{2}\right)\mathbb{Z}$, with $v^2 + d \equiv 0 \pmod{16}$, and assume $(d, 3) = 1$. If \mathcal{A} represents any ideal class in the order R_{-d} , the number $2f_w(\mathcal{A})/f_w(\mathcal{T}\mathcal{A})^2$ is the Y -coordinate of an integral point (X, Y) on the curve

$$Y^4 = X(X^2 + 4)$$

with coordinates in Ω_f . The corresponding X -coordinate is the negative conjugate $X = -\beta_c = -\beta^{\tau_c^{-1}}$ of β , where $\tau_c = (\Omega_f/K, R_K \mathfrak{c})$ with $\mathfrak{c} \in \mathcal{A}$.

We relate the numbers α_1 and β that we study to the Weber singular moduli of Yui and Zagier by showing that the number $4\alpha_1/\beta^2 = 4\zeta_8^j \alpha/\beta^2$ is the 6-th power of the unit $f_w(Q_2)$ considered by Yui and Zagier, where $Q_2(x, y) = 2x^2 - vxy + \left(\frac{v^2+d}{8}\right)y^2$, whether or not 3 divides d . Furthermore, we show that if $3 \mid d$ and $Q_1(x, y) = x^2 - vxy + \left(\frac{v^2+d}{4}\right)y^2$, then the product $f_w(Q_1)f_w(Q_2)$ of the Weber singular moduli for the forms Q_1 and Q_2 is a generator of the ring class field Ω_f over K (for $d > 15$), although neither of the numbers $f_w(Q_1), f_w(Q_2)$ lies in Ω_f . Examples show that this number has small height and relatively small discriminant, as is the case for the numbers $f_w(Q_1)$ when $(d, 3) = 1$. This extends the discussion in the paper [36] for the case $3 \mid d$ and gives a systematic procedure for producing generators with small height and discriminant in this case. We give a number of examples of the minimal polynomials of these generators in Sections 11 and 12.

In our discussion we rely mostly on basic facts from class field theory and the theory of complex multiplication, as well as a few facts about the Dedekind eta-function and the Schläfli functions, which can be found in Weber's treatise [35]. (See

also [32], [33], and [36].) In particular, we do not use any of the advanced theory of Heegner points in our discussion. We take a mostly algebraic approach in Sections 3-9; this is advantageous when we turn to analytic representations of our solutions in Sections 10 and 11. This combined approach leads to the proof of Theorem 1.4 (see Theorems 10.4 and 10.5) and to the proof that the numbers $f_w(Q_1)f_w(Q_2)$ above have degree $h(-d)$ over \mathbb{Q} in Theorem 11.1.

2 Weber's functions and the j -invariant.

In this section we will use classical functions considered by Weber [35] to recognize specific values of the j -function as j -invariants of certain elliptic curves in Tate normal form.

We start with Weber's functions

$$\begin{aligned} x &= \mathfrak{f}_2(w)^8 = 16 \frac{\eta(2w)^8}{\eta(w)^8}, & x_0 &= \mathfrak{f}_1(w)^8 = \frac{\eta(w/2)^8}{\eta(w)^8}, \\ x_1 &= -\mathfrak{f}(w)^8 = e^{2\pi i/3} \frac{\eta((w+1)/2)^8}{\eta(w)^8} = \frac{\eta((w+3)/2)^8}{\eta(w)^8}, \end{aligned}$$

where $\eta(z)$ is the Dedekind eta-function. (See [35, p. 250], [30], [31], and Section 10 for the definitions of the Schläfli functions $\mathfrak{f}, \mathfrak{f}_1, \mathfrak{f}_2$.) These functions satisfy the equation

$$X^3 - X\gamma_2(w) + 16 = 0, \quad \gamma_2(w) = j(w)^{1/3}. \quad (2.1)$$

Thus, we have

$$j(w) = \gamma_2(w)^3 = \left(\frac{x^3 + 16}{x} \right)^3. \quad (2.2)$$

Replacing w by $w/2$ transforms the function x into $16/x_0$; it follows that

$$16^2 - x_0^2 \gamma_2\left(\frac{w}{2}\right) + x_0^3 = 0, \quad (2.3)$$

and therefore

$$j\left(\frac{w}{2}\right) = \gamma_2\left(\frac{w}{2}\right)^3 = \frac{(x_0^3 + 16^2)^3}{x_0^6}. \quad (2.4)$$

Now set

$$y(w) = \frac{\eta(w/4)^4}{\eta(w)^4} = \mathfrak{f}_1(w/2)^4 \mathfrak{f}_1(w)^4,$$

so that $y^2 = x_0(w/2)x_0(w)$. Replace w once again by $w/2$ in (2.1) for the function x_0 in place of x and use $x_0^2\gamma_2(w/2) = 16^2 + x_0^3$ from (2.3). This gives

$$\begin{aligned} 0 &= y^6 - y^2 x_0^2 \gamma_2\left(\frac{w}{2}\right) + 16x_0^3 = y^6 - y^2(16^2 + x_0^3) + 16x_0^3 \\ &= (y^2 - 16)(y^4 + 16y^2 - x_0^3). \end{aligned}$$

It follows that $x_0^3 = y^4 + 16y^2$; putting this into (2.2) (with x_0 for x) and (2.4) gives

$$j(w) = \frac{(y^4 + 16y^2 + 16)^3}{y^4 + 16y^2}, \quad j\left(\frac{w}{2}\right) = \frac{(y^4 + 16y^2 + 256)^3}{(y^4 + 16y^2)^2}. \quad (2.5)$$

Putting $w/2$ for w in (2.3) leads similarly to the equation

$$16^2 x_0^3 - y^4 x_0 \gamma_2\left(\frac{w}{4}\right) + y^6 = 0.$$

Rearranging and cubing gives

$$\begin{aligned} 16^2(y^4 + 16y^2) + y^6 &= y^4 x_0 \gamma_2\left(\frac{w}{4}\right) \\ (y^6 + 16^2 y^4 + 16^3 y^2)^3 &= y^{12}(y^4 + 16y^2)j\left(\frac{w}{4}\right). \end{aligned}$$

Hence,

$$j\left(\frac{w}{4}\right) = \frac{(y^4 + 256y^2 + 4096)^3}{y^8(y^2 + 16)}. \quad (2.6)$$

Thus, $j(w/4)$ arises from $j(w)$ by the substitution $y \rightarrow 16/y$.

Now we set

$$\alpha = \zeta_8^{-1} \sqrt{y(w)} = \zeta_8^{-1} \frac{\eta(w/4)^2}{\eta(w)^2}, \quad \zeta_8 = e^{2\pi i/8}, \quad (2.7)$$

so that $\alpha^2 = -iy(w)$. From (2.5) and (2.6) we obtain

$$j(w) = \frac{(\alpha^8 - 16\alpha^4 + 16)^3}{\alpha^8 - 16\alpha^4}, \quad j\left(\frac{w}{2}\right) = \frac{(\alpha^8 - 16\alpha^4 + 256)^3}{\alpha^8(\alpha^4 - 16)^2}, \quad (2.8)$$

$$j\left(\frac{w}{4}\right) = \frac{(\alpha^8 - 256\alpha^4 + 4096)^3}{\alpha^{16}(16 - \alpha^4)}. \quad (2.9)$$

From the calculations of [27, pp.253-254] we see that $j(w)$ is the j -invariant of the elliptic curve

$$E_1(\alpha) : Y^2 + XY + \frac{1}{\alpha^4}Y = X^3 + \frac{1}{\alpha^4}X^2, \quad (2.10)$$

which is the Tate normal form for a curve with a point of order $n = 4$; and $j(w/2)$ is the j -invariant of the elliptic curve

$$E_2(\alpha) : Y^2 + XY + \frac{2}{\alpha^4}Y = X^3 + \frac{4}{\alpha^4}X^2 - \frac{1}{\alpha^8}, \quad (2.11)$$

which is 2-isogenous to $E_1(\alpha)$ by the map $\psi = (\psi_1, \psi_2) : E_1(\alpha) \rightarrow E_2(\alpha)$ with

$$\psi_1(X, Y) = \frac{X^2}{X + b}, \quad \psi_2(X, Y) = \frac{-b^2}{X + b} + \frac{X(X + 2b)Y}{(X + b)^2}, \quad b = \frac{1}{\alpha^4}.$$

Still with $b = 1/\alpha^4$, the isogeny

$$X_1 = \phi_1(X, Y) = \frac{X^2 - b}{X + 4b},$$

$$Y_1 = \phi_2(X, Y) = \frac{bX^2 + (b - 8b^2)X + 3b^2 - 32b^3}{(X + 4b)^2} + \frac{X^2 + 8bX + b}{(X + 4b)^2}Y$$

maps $E_2(\alpha)$ to the curve

$$E_3(\alpha) : Y_1^2 + X_1Y_1 + \frac{4}{\alpha^4}Y_1 = X_1^3 + \frac{16}{\alpha^4}X_1^2 + \frac{6}{\alpha^4}X_1 + \frac{\alpha^4 - 4}{\alpha^8}, \quad (2.12)$$

and the j -invariant of this curve is the expression $j(w/4)$ in (2.9). It is not difficult to verify that the isogeny $(E_1(\alpha) \rightarrow E_3(\alpha))$ is cyclic, and therefore represents a point on the modular curve $X_0(4)$.

3 Determining the field of definition.

Now let $w = (v + \sqrt{-d})/2$, where $-d \equiv 1 \pmod{8}$ is the discriminant of the quadratic field $K = \mathbb{Q}(\sqrt{-d})$ and v is an odd integer satisfying $v^2 \equiv -d \pmod{16}$. Then $(2, w) = \wp_2$ is a prime divisor of 2 in the ring of integers R_K of K with the basis $\{2, w\}$ and \wp_2^2 has the basis $\{4, w\}$. It follows that $j = j(w)$, $k = j(w/2)$ and $l = j(w/4)$ are roots of the class equation $H_{-d}(x) = 0$ and elements of the Hilbert class field $\Sigma = \mathbb{Q}(\sqrt{-d}, j(w))$ of K . (For the application in Sections 10 and 11, we note that we could just as well take $w = (v + \sqrt{-d})/(2a)$, where $16a \mid (v^2 + d)$, in which case $\{a, aw\}$ is a basis for an ideal \mathfrak{a} and $\{4a, aw\}$ is a basis for $\wp_2^2 \mathfrak{a}$.)

From (2.8) and (2.9) we see that $t = \alpha^4$ is a root of three sixth degree equations over Σ :

$$\begin{aligned} f(x) &= (x^2 - 16x + 16)^3 - j(x^2 - 16x), \\ g(x) &= (x^2 - 16x + 256)^3 - k(x^2 - 16x)^2, \\ h(x) &= (x^2 - 256x + 4096)^3 - lx^4(16 - x). \end{aligned}$$

We would like to show that the ideal $I = (f(x), g(x), h(x))$ in $R_\Sigma[x]$ contains a non-constant linear polynomial. We perform the first two steps of the Euclidean algorithm on $f(x)$ and $g(x)$. First we set

$$\begin{aligned} q_1(x) &= g(x) - f(x) = (720 - k)x^4 + (-23040 + 32k)x^3 + (j + 380160 - 256k)x^2 \\ &\quad + (-3133440 - 16j)x + 16773120. \end{aligned}$$

Next, take the remainder of $f(x)$ with respect to $q_1(x)$:

$$\begin{aligned} (k - 720)^2 f(x) - \left((720 - k)x^2 + (-11520 + 16k)x - 48k - j - 161280 \right) q_1(x) = \\ q_2(x) = (768k^2 - k^2j + 25067520k + 1488kj - 161280j + 19906560000 + j^2)x^2 \\ + (-12288k^2 + 16k^2j - 401080320k - 23808kj + 2580480j - 318504960000 - 16j^2)x \\ + 4096k^2 + 799211520k + 2707292160000 + 16773120j. \end{aligned}$$

Setting

$$a_2(j, k) = 768k^2 - k^2j + 25067520k + 1488kj - 161280j + 19906560000 + j^2,$$

we calculate the remainder of

$$q_3(x) = h(x) - g(x) = (-720 + l)x^5 + (k - 16l + 207360)x^4 + (-32k - 23040000)x^3 \\ + (256k + 855244800)x^2 - 12881756160x + 68702699520$$

with respect to $q_2(x)$ and find

$$a_2(j, k)^2 q_3(x) - u(x) q_2(x) = a_3(j, k, l)x + b_3(j, k, l),$$

where $u(x)$ is cubic and

$$a_3(j, k, l) = -2^{20}(9331200000 - 226800j + j^2 + 21945600k + 1488kj + 752k^2 - k^2j) \\ \times (4095j^2 + 6093360kj - 663390000j - 4095k^2j + 4095jl + 102511008000k \\ + 660960000l + 3144240k^2 + 195120kl + k^2l + 81041472000000) \\ = -2^{20}A_1(j, k)A_2(j, k, l).$$

We take the resultant of the first factor of $a_3(j, k, l)$ and the modular equation

$$\Phi_2(j, k) = k^3 + 1488k^2j - k^2j^2 - 162000k^2 + 8748000000k + 1488j^2k + 40773375kj \\ + j^3 - 162000j^2 + 8748000000j - 157464000000000$$

with respect to k and find

$$\text{Res}_k(A_1(j, k), \Phi_2(j, k)) = -2^{32}(j - 54000)(j + 3375)^2(j^2 + 191025j - 121287375)^2 \\ = -2^{32}H_{-12}(j)H_{-7}(j)^2H_{-15}(j)^2.$$

Next we take the resultant of the second factor of $a_3(j, k, l)$ and $\Phi_2(k, l)$ with respect to l and then take the resultant of that polynomial and $\Phi_2(j, k)$ with respect to k :

$$\begin{aligned}
& \text{Res}_k(\Phi_2(j, k), \text{Res}_l(\Phi_2(k, l), A_2(j, k, l))) = \\
& -2^{85}(-574852565088000000 - 1645137216000j - 4519800j^2 + 2047j^3) \\
& \times (j - 16581375)^2(j^2 - 37018076625j + 153173312762625)^2 \\
& \times (j + 3375)^6(j^2 + 191025j - 121287375)^6 \\
& = -2^{85}A_3(j)H_{-28}(j)^2H_{-60}(j)^2H_{-7}(j)^6H_{-15}(j)^6.
\end{aligned}$$

Now the roots of the cubic $A_3(j)$ are not algebraic integers, and $j = j(w)$ is only a root of the class equation $H_{-d}(x)$. Hence, if $d \neq 7, 15$, the above resultants cannot be zero, so that $a_3(j, k, l) \neq 0$. Thus the ideal I contains a non-constant linear polynomial in $R_\Sigma[x]$, and it follows, since $t = \alpha^4$ is a root of each of the polynomials f, g, h , that $\alpha^4 \in \Sigma$. We summarize this in

Proposition 3.1. Let $-d \equiv 1 \pmod{8}$ be the discriminant of the quadratic field $K = \mathbb{Q}(\sqrt{-d})$. If

$$w = \frac{v + \sqrt{-d}}{2}, \text{ with } v^2 \equiv -d \pmod{16},$$

then the number $-\alpha^4 = \frac{\eta(w/4)^8}{\eta(w)^8}$ lies in the Hilbert class field Σ of K .

In the case $d = 7$ we can use (2.8) and (2.9) along with $H_{-7}(x) = x + 3375$ to compute directly that

$$t = \alpha^4 = \frac{31 \pm 3\sqrt{-7}}{2}.$$

Similarly, when $d = 15$ we can use $H_{-15}(x) = x^2 + 191025x - 121287375$ to compute that $t = \alpha^4$ is a root of the polynomial

$$r_{15}(t) = t^4 - 17t^3 + 33t^2 - 4352t + 65536.$$

This polynomial has degree $4 = 2h(-15)$, so it is easy to see that its roots lie in the Hilbert class field of $\mathbb{Q}(\sqrt{-15})$. This verifies the proposition in the two remaining cases $d = 7, 15$.

Proposition 3.1 also follows from Theorem 6.6.4 in Schertz's monograph [32, p. 159]. As a corollary we have

Proposition 3.2. With d and α as in Proposition 3.1, the curves $E_1(\alpha)$, $E_2(\alpha)$, and $E_3(\alpha)$ in (2.10)-(2.12) have complex multiplication by the ring of integers R_K of $K = \mathbb{Q}(\sqrt{-d})$ and are defined over the Hilbert class field Σ of K .

If $-d = -f^2 d_1$ with odd $f > 1$ and d_1 square-free, then there is an analogous result for the order R_{-d} of discriminant $-d$ in K , since the ideal $\wp_{2,-d} = \wp_2 \cap R_{-d}$ in R_{-d} of elements divisible by $\wp_2 = (2, w)$ has the \mathbb{Z} -basis $\{2, (v + \sqrt{-d})/2\}$ in R_{-d} (cf. [10, pp. 19-20]). With α defined as in (2.7), for $w = (v + \sqrt{-d})/2$ and $v^2 \equiv -d \pmod{16}$, all the same formulas hold, and the numbers $j(w)$, $j(w/2)$, $j(w/4)$ are roots of $H_{-d}(x) = 0$. Each of these numbers generates the ring class field Ω_f of conductor f over K , and exactly as before we obtain:

Proposition 3.3. If $-d = -f^2 d_1$ with odd f and square-free $d_1 \equiv 7 \pmod{8}$, α as in (2.7), and $w = (v + \sqrt{-d})/2$ as above, the curves $E_1(\alpha)$, $E_2(\alpha)$, and $E_3(\alpha)$ in (2.10)-(2.12) have complex multiplication by the order R_{-d} of $K = \mathbb{Q}(\sqrt{-d_1})$ and are defined over the ring class field Ω_f of conductor f over K .

This proposition shows that for the given d and α , the isogenies $(E_1(\alpha) \rightarrow E_3(\alpha))$ are examples of *Heegner points* on the curve $X_0(4)$. See [2] and [16].

4 Points of order 2 and 4 on E_1 .

For the application in Section 6 we derive formulas for all the points of order 4 on the curve $E_1(\alpha)$ in terms of the number β defined by the diophantine condition that $(X, Y) = (\alpha, \beta)$ lies on the curve

$$Fer_4 : 16X^4 + 16Y^4 = X^4 Y^4.$$

We let $E_1 = E_1(\alpha)$ and set $b = 1/\alpha^4$, so that

$$E_1 : Y^2 + XY + bY = X^3 + bX^2, \quad b = \frac{1}{\alpha^4}. \quad (4.1)$$

The doubling formula for the X -coordinate of a point $P = (x, y)$ on E_1 is

$$X(2P) = \frac{x^4 - bx^2 - 2b^2x - b^3}{(x + b)(4x^2 + x + b)}, \quad x = X(P).$$

Iterating this formula, we find that the denominator of $X(4P)$ is the polynomial

$$(x + b)(4x^2 + x + b)x^2(x + 2b)^2(2x^4 + x^3 + 3bx^2 + 4b^2x + 2b^3)^2.$$

The roots of the first two factors are the X -coordinates of the points of order 2, which are given by $X = -b = -1/\alpha^4$ and

$$X = \frac{-1}{8} \pm \frac{1}{8}\sqrt{1 - 16b} = \frac{-1}{8} \pm \frac{1}{8}\sqrt{\frac{\alpha^4 - 16}{\alpha^4}} = \frac{-1}{8} \pm \frac{1}{8}\frac{4}{\beta^2} = \frac{-\beta^2 \pm 4}{8\beta^2}, \quad (4.2)$$

on using the defining condition $16\alpha^4 + 16\beta^4 = \alpha^4\beta^4$ for β .

The X -coordinates of the points of order 4 on E_1 are solutions of the equation

$$x(x + 2b)(2x^4 + x^3 + 3bx^2 + 4b^2x + 2b^3) = 0.$$

The first factor gives the points $(0, 0)$ and $(0, -b)$. The second factor gives the points

$$x = -2b, \quad y = \frac{b}{2}(1 \pm \sqrt{1 - 16b}).$$

Using $16\alpha^4 + 16\beta^4 = \alpha^4\beta^4$ we can write the y -coordinates of these points in the form

$$y = \frac{b}{2}(1 \pm \sqrt{1 - 16b}) = \frac{b}{2} \left(1 \pm \sqrt{\frac{\alpha^4 - 16}{\alpha^4}} \right) = \frac{b}{2} \left(1 \pm \frac{4}{\beta^2} \right). \quad (4.3)$$

Finding the remaining eight points of order 4 requires solving the quartic equation $2x^4 + x^3 + 3bx^2 + 4b^2x + 2b^3 = 0$, which we do by introducing the indeterminate z and completing the square:

$$\left(x^2 + \frac{1}{4}x + z \right)^2 = \frac{1}{16}(32z + 1 - 24b)x^2 + \frac{1}{2}(z - 4b^2)x + z^2 - b^3.$$

We choose $z = \frac{b}{4}$ to make the right hand side of the last equation a square, and find that

$$\left(x^2 + \frac{1}{4}x + \frac{b}{4}\right)^2 = \left(\frac{1}{4}\sqrt{1-16b}(x+b)\right)^2.$$

This leads to the quadratic equations

$$x^2 + \frac{1}{4}(1 \pm \sqrt{1-16b})x + \frac{b}{4}(1 \pm \sqrt{1-16b}) = 0,$$

whose solutions are

$$\begin{aligned} x = & -\frac{1}{8} \left(1 - \sqrt{1-16b} \pm \sqrt{2}\sqrt{1-16b - (1-8b)\sqrt{1-16b}} \right), \\ & -\frac{1}{8} \left(1 + \sqrt{1-16b} \pm \sqrt{2}\sqrt{1-16b + (1-8b)\sqrt{1-16b}} \right). \end{aligned} \quad (4.4)$$

The corresponding values of y can be found by solving $y^2 + xy + by = x^3 + bx^2$ for y :

$$y = -\frac{1}{2} \left(x + b \pm \sqrt{(x+b)(4x^2 + x + b)} \right). \quad (4.5)$$

We shall write these solutions in terms of the solution (α, β) of Fer_4 . Note that

$$b = \frac{1}{\alpha^4} = \frac{\beta^4 - 16}{16\beta^4} = \left(\frac{\beta + 2i}{2\beta}\right) \left(\frac{\beta - 2}{2\beta}\right) \left(\frac{\beta - 2i}{2\beta}\right) \left(\frac{\beta + 2}{2\beta}\right) = \beta_1\beta_2\beta_3\beta_4$$

where $i = \sqrt{-1}$ and

$$\beta_n = \frac{\beta + 2i^n}{2\beta}, \quad 1 \leq n \leq 4. \quad (4.6)$$

Next, note that

$$1 - 8b = \frac{\beta^4 + 16}{2\beta^4}, \quad 1 - 16b = \frac{16}{\beta^4}. \quad (4.7)$$

We will take $\sqrt{1-16b} = \frac{4}{\beta^2}$, since in all the formulas below this expression appears with both plus and minus signs. For the points of order 2 we have from (4.2):

$$X = \frac{-\beta^2 + 4}{8\beta^2} = -\frac{1}{2} \left(\frac{\beta - 2}{2\beta} \right) \left(\frac{\beta + 2}{2\beta} \right) = -\frac{1}{2}\beta_2\beta_4,$$

and the corresponding Y -coordinate is

$$\begin{aligned} Y &= \frac{1}{16}(1 - 8b - \sqrt{1 - 16b}) = \frac{1}{16} \left(\frac{\beta^4 + 16}{2\beta^4} - \frac{4}{\beta^2} \right) = \frac{1}{2} \left(\frac{\beta^2 - 4}{4\beta^2} \right)^2 \\ &= \frac{1}{2} \left(\frac{\beta - 2}{2\beta} \right)^2 \left(\frac{\beta + 2}{2\beta} \right)^2 = \frac{1}{2}\beta_2^2\beta_4^2. \end{aligned}$$

Similarly, taking the minus sign in (4.2) leads to the coordinates

$$X = \frac{1}{8}(-1 - \sqrt{1 - 16b}) = -\frac{1}{2}\beta_1\beta_3, \quad Y = \frac{1}{16}(1 - 8b + \sqrt{1 - 16b}) = \frac{1}{2}\beta_1^2\beta_3^2.$$

Thus the points of order 2 on E_1 are given by

$$(-\beta_1\beta_2\beta_3\beta_4, 0), \left(-\frac{1}{2}\beta_1\beta_3, \frac{1}{2}\beta_1^2\beta_3^2 \right), \left(-\frac{1}{2}\beta_2\beta_4, \frac{1}{2}\beta_2^2\beta_4^2 \right). \quad (4.8)$$

For the Y -coordinates of the points of order 4 in (4.3) we have

$$\frac{b}{2}(1 + \sqrt{1 - 16b}) = \frac{b}{2} \left(1 + \frac{4}{\beta^2} \right) = 2b \left(\frac{\beta + 2i}{2\beta} \right) \left(\frac{\beta - 2i}{2\beta} \right) = 2\beta_1^2\beta_2\beta_3^2\beta_4.$$

Similarly, $\frac{b}{2}(1 - \sqrt{1 - 16b}) = 2\beta_1\beta_2^2\beta_3\beta_4^2$. This gives the following points of order 4:

$$(0, 0), (0, -\beta_1\beta_2\beta_3\beta_4), (-2\beta_1\beta_2\beta_3\beta_4, 2\beta_1^2\beta_2\beta_3^2\beta_4), (-2\beta_1\beta_2\beta_3\beta_4, 2\beta_1\beta_2^2\beta_3\beta_4^2). \quad (4.9)$$

The simplifications that occur for the other eight points of order 4 are remarkable. First notice that

$$\begin{aligned} 1 - 16b - (1 - 8b)\sqrt{1 - 16b} &= \frac{16}{\beta^4} - \left(\frac{\beta^4 + 16}{2\beta^4} \right) \left(\frac{4}{\beta^2} \right) \\ &= -\frac{2}{\beta^6}(-8\beta^2 + \beta^4 + 16) = -2 \left(\frac{\beta^2 - 4}{\beta^3} \right)^2. \end{aligned}$$

Similarly, $1 - 16b + (1 - 8b)\sqrt{1 - 16b} = 2 \left(\frac{\beta^2 + 4}{\beta^3} \right)^2$. We will work out the corresponding expression completely for

$$x = -\frac{1}{8} \left(1 - \sqrt{1 - 16b} + \sqrt{2} \sqrt{1 - 16b - (1 - 8b)\sqrt{1 - 16b}} \right),$$

the other cases being similar. Making use of the above substitutions yields

$$\begin{aligned} x &= -\frac{1}{8} \left(1 - \frac{4}{\beta^2} + \sqrt{2}\sqrt{2}i \left(\frac{\beta^2 - 4}{\beta^3} \right) \right) = -\frac{1}{8\beta^3} (\beta^3 + 2i\beta^2 - 4\beta - 8i) \\ &= -\frac{1}{8\beta^3} (\beta + 2i)(\beta - 2)(\beta + 2) = -\beta_1\beta_2\beta_4. \end{aligned}$$

Now we use (4.5) to find the corresponding Y -coordinates. Using

$$-1 + \beta_3 = -\beta_1, \quad 4\beta_2\beta_4 - 1 = -\frac{4}{\beta^2}$$

we can simplify the expression under the square root in (4.5):

$$\begin{aligned} x + b &= -\beta_1\beta_2\beta_4 + \beta_1\beta_2\beta_3\beta_4 = \beta_1\beta_2\beta_4(-1 + \beta_3) = -\beta_1^2\beta_2\beta_4, \\ 4x^2 + (x + b) &= 4\beta_1^2\beta_2^2\beta_4^2 - \beta_1^2\beta_2\beta_4 = \beta_1^2\beta_2\beta_4(4\beta_2\beta_4 - 1) = -\frac{4\beta_1^2\beta_2\beta_4}{\beta^2}. \end{aligned}$$

We now have

$$\begin{aligned} -\frac{1}{2} \left(x + b \pm \sqrt{(x + b)(4x^2 + x + b)} \right) &= -\frac{1}{2} \left(-\beta_1^2\beta_2\beta_4 \pm \frac{2\beta_1^2\beta_2\beta_4}{\beta} \right) \\ &= \frac{\beta_1^2\beta_2\beta_4}{2} \left(1 \mp \frac{2}{\beta} \right) = \beta_1^2\beta_2\beta_4 \left(\frac{\beta \mp 2}{2\beta} \right). \end{aligned}$$

Depending on the \mp sign, the last expression is either $\beta_1^2\beta_2^2\beta_4$ or $\beta_1^2\beta_2\beta_4^2$. These are the two values of y that correspond to $x = -\beta_1\beta_2\beta_4$. In a similar manner, we obtain the other six points of order 4. We summarize these results:

Proposition 4.1. (See [26].) If (α, β) is a point on the curve Fer_4 and β_n is defined by (4.6), the 4-torsion points on the elliptic curve

$$E_1(\alpha) : Y^2 + XY + bY = X^3 + bX^2 \quad \text{with} \quad b = \frac{1}{\alpha^4} = \beta_1\beta_2\beta_3\beta_4$$

and base point O , are given by

$$\begin{aligned} & O, (-\beta_1\beta_2\beta_3\beta_4, 0), \left(-\frac{1}{2}\beta_1\beta_3, \frac{1}{2}\beta_1^2\beta_3^2\right), \left(-\frac{1}{2}\beta_2\beta_4, \frac{1}{2}\beta_2^2\beta_4^2\right), \\ & (0, 0), (0, -\beta_1\beta_2\beta_3\beta_4), (-2\beta_1\beta_2\beta_3\beta_4, 2\beta_1^2\beta_2\beta_3^2\beta_4), (-2\beta_1\beta_2\beta_3\beta_4, 2\beta_1\beta_2^2\beta_3\beta_4^2), \\ & (-\beta_1\beta_2\beta_3, \beta_1^2\beta_2^2\beta_3), (-\beta_1\beta_2\beta_3, \beta_1\beta_2^2\beta_3^2), (-\beta_1\beta_2\beta_4, \beta_1^2\beta_2^2\beta_4), (-\beta_1\beta_2\beta_4, \beta_1\beta_2\beta_4^2), \\ & (-\beta_1\beta_3\beta_4, \beta_1^2\beta_3\beta_4^2), (-\beta_1\beta_3\beta_4, \beta_1\beta_3^2\beta_4^2), (-\beta_2\beta_3\beta_4, \beta_2^2\beta_3^2\beta_4), (-\beta_2\beta_3\beta_4, \beta_2\beta_3^2\beta_4^2). \end{aligned}$$

Remarks. 1. The correctness of the above formulas can be verified by noting that if β is replaced by $i\beta$, then β_n in (4.6) becomes β_{n-1} . Thus, the last eight points are permuted among themselves in two orbits of order 4 on iterating the map $\beta \rightarrow i\beta$.

2. From the definition (4.6) of β_n we have $2\beta_n - 1 = \frac{2i^n}{\beta}$, so $(2\beta_n - 1)^4 = \frac{16}{\beta^4} = 1 - 16b$.

Note that for any solution (α, β) of Fer_4 we have $\beta^4 \neq 16$, so that none of the β_n are zero. Hence, we have

Proposition 4.2. The non-zero coordinates of the points in $E_1[4]$ lie in the multiplicative group generated by -1 , 2 , and β_n , for $1 \leq n \leq 4$.

Remark. In [26], the first author shows that the points of order 8 on E_1 are also expressible in terms of quantities defined in terms of a solution (α, β) of Fer_4 . The resulting expressions are similar to those for the points listed in Proposition 4.1, in that the X - and Y -coordinates are given as products of certain quantities that depend algebraically on α and β .

5 Points of order 4 on E_3 .

We shall now find linear fractional expressions for the X -coordinates of some of the points of order 4 on the curve

$$E_3(\beta) : Y^2 + XY + 4bY = X^3 + 16bX^2 + 6bX + b - 4b^2, \quad b = \frac{1}{\beta^4}. \quad (5.1)$$

On this curve we have the doubling formula

$$X(2P) = \frac{x^4 - 16bx^2 - 8bx - b}{(4x + 1)(x^2 + 16bx + 4b)}, \quad x = X(P). \quad (5.2)$$

We set

$$\gamma_n = \frac{\alpha}{\alpha + \beta \zeta_8^{2n-1}}, \quad 1 \leq n \leq 4, \quad (5.3)$$

and note that

$$b = \frac{1}{\beta^4} = \frac{\alpha^4}{16(\alpha^4 + \beta^4)} = \frac{1}{16} \left(\frac{\alpha}{\alpha + \beta \zeta_8} \right) \left(\frac{\alpha}{\alpha + \beta \zeta_8^3} \right) \left(\frac{\alpha}{\alpha + \beta \zeta_8^5} \right) \left(\frac{\alpha}{\alpha + \beta \zeta_8^7} \right).$$

We now iterate (5.2) and find that the X -coordinates of the points of order 4 are zeros of the polynomial

$$x(2x + 1)(x^4 + 32bx^3 + 24bx^2 + 8bx + b).$$

We shall focus on the roots of the quartic factor. Treating this factor as we did in Section 4, we complete the square twice to obtain

$$(x^2 + 16bx + z)^2 = (256b^2 - 24b + 2z)x^2 + (32bz - 8b)x + z^2 - b$$

with the indeterminate z . Setting $z = 4b$, the right side becomes a square:

$$(x^2 + 16bx + 4b)^2 = b(16b - 1)(4x + 1)^2;$$

and using $b(16b - 1) = -\frac{16}{\alpha^4 \beta^4}$ we have the two quadratic equations

$$(x^2 + 16bx + 4b) \pm \frac{4i}{\alpha^2 \beta^2} (4x + 1) = 0,$$

or

$$x^2 + 16 \left(b \pm \frac{i}{\alpha^2 \beta^2} \right) x + 4 \left(b \pm \frac{i}{\alpha^2 \beta^2} \right) = 0.$$

Note that

$$\begin{aligned} b + \frac{i}{\alpha^2\beta^2} &= \frac{1}{\beta^4} \left(\frac{\alpha^2 + \beta^2 i}{\alpha^2} \right) = b \left(\frac{\alpha + \beta\zeta_8^3}{\alpha} \right) \left(\frac{\alpha + \beta\zeta_8^7}{\alpha} \right) \\ &= \frac{1}{16} \gamma_1 \gamma_2 \gamma_3 \gamma_4 \cdot \frac{1}{\gamma_2 \gamma_4} = \frac{1}{16} \gamma_1 \gamma_3. \end{aligned}$$

Similarly, $b - \frac{i}{\alpha^2\beta^2} = \frac{1}{16} \gamma_2 \gamma_4$. Thus, the two quadratic equations become

$$x^2 + \gamma_1 \gamma_3 x + \frac{1}{4} \gamma_1 \gamma_3 = 0, \quad x^2 + \gamma_2 \gamma_4 x + \frac{1}{4} \gamma_2 \gamma_4 = 0.$$

Using the relations $\gamma_1 - 2\gamma_1\gamma_3 + \gamma_3 = 0$ and $\gamma_2 - 2\gamma_2\gamma_4 + \gamma_4 = 0$ it can be checked directly that the roots of the first equation are $x = -\frac{1}{2}\gamma_1, -\frac{1}{2}\gamma_3$, and the roots of the second are $x = -\frac{1}{2}\gamma_2, -\frac{1}{2}\gamma_4$. This proves

Proposition 5.1. The values

$$x = -\frac{1}{2}\gamma_n = -\frac{1}{2} \frac{\alpha}{\alpha + \beta\zeta_8^{2n-1}}, \quad 1 \leq n \leq 4,$$

are the X -coordinates of points of order 4 on the curve $E_3 = E_3(\beta)$.

6 Solutions of the Fermat equation.

We can now put together the insights we have gained in Sections 2-5. When $-d$ is square-free, we know that the curve $E_1(\alpha)$ has complex multiplication by the maximal order R_K of K and is defined over Σ . Torsion points on this curve of order 2 or 4 will therefore generate low degree abelian extensions of K . The form of the X -coordinates of points of order 2 or 4 will allow us to locate simple functions of α and β in the field Σ .

The Weierstrass normal form of the curve $E_1(\alpha)$ in (2.10) or (4.1) is

$$Y_1^2 = 4X_1^3 - g_2X_1 - g_3 = 4X_1^3 - \left(\frac{16b^2 - 16b + 1}{12} \right) X_1 + \frac{64b^3 + 120b^2 - 24b + 1}{216},$$

$$\text{with } Y_1 = 2Y + X + b, \quad X_1 = X + \frac{4b+1}{12}, \quad b = \frac{1}{\alpha^4}.$$

From the theory of complex multiplication [33, p. 135] the ray class field Σ_2 over K with conductor 2 is generated over K by the values $j(E_1)$ and

$$h(P) = \frac{g_2 g_3}{\Delta} \left(X(P) + \frac{4b+1}{12} \right), \quad (6.1)$$

where P runs over the points of order 2 on E_1 . However, the Euler- φ function $\varphi(2) = \varphi(\wp_2)\varphi(\wp'_2) = 1$ in K , so that $\Sigma_2 = \Sigma$. By Proposition 3.1 we know that g_2, g_3 and $\Delta = b^4(1 - 16b)$ lie in Σ . Letting P be the last point in (4.8), we deduce that $X(P) = -(\beta^2 - 4)/(8\beta^2)$ lies in Σ , and it follows that $\beta^2 \in \Sigma$.

We now do the same for the curve

$$E_3(\beta) : Y^2 + XY + 4bY = X^3 + 16bX^2 + 6bX + b - 4b^2, \quad b = \frac{1}{\beta^4},$$

whose j -invariant we know to be equal to

$$\frac{(\beta^8 - 256\beta^4 + 4096)^3}{\beta^{16}(16 - \beta^4)} = \frac{(\alpha^8 - 256\alpha^4 + 4096)^3}{\alpha^{16}(16 - \alpha^4)} = j\left(\frac{w}{4}\right), \quad (6.2)$$

by virtue of (2.9) and the defining condition for β . Thus, $E_3(\beta)$ has complex multiplication by R_K . The Weierstrass normal form of $E_3(\beta)$ has coefficients in Σ , so once again the X -coordinates of points of order 2 on $E_3(\beta)$ must lie in Σ . From (5.2) these coordinates are

$$X = -1/4, \quad -8b \pm 2\sqrt{16b^2 - b} = -\frac{8}{\beta^4} \pm 2\sqrt{\frac{16 - \beta^4}{\beta^8}} = -\frac{8}{\beta^4} \pm \frac{8i}{\alpha^2\beta^2}.$$

It follows that $i\alpha^2\beta^2 \in \Sigma$, whence we have that $i\alpha^2 \in \Sigma$. From Proposition 3.1 we now obtain

Proposition 6.1. With notation as in Proposition 3.1, the number

$$i\alpha^2 = y(w) = \frac{\eta(w/4)^4}{\eta(w)^4}$$

lies in the Hilbert class field Σ , along with the number β^2 defined by $16\alpha^4 + 16\beta^4 = \alpha^4\beta^4$.

Remark. This proposition improves upon a result of Schertz [31], according to which $(\eta(w/4)/\eta(w))^8 \in \Sigma$. See also [32, p. 159].

Consider next the point $P = (-\beta_1\beta_2\beta_3, \beta_1^2\beta_2^2\beta_3)$, which has order 4 on $E_1(\alpha)$. Its X -coordinate,

$$X(P) = -\beta_1\beta_2\beta_3 = -\frac{\beta + 2i}{2\beta} \frac{\beta - 2}{2\beta} \frac{\beta - 2i}{2\beta} = -\frac{\beta^2 + 4}{4\beta^2} \frac{\beta - 2}{2\beta},$$

lies in the field Σ_4 . It is not hard to see that $\Sigma_4 = \Sigma(i)$. First, we have $\varphi(4) = \varphi(\wp_2^2\wp_2'^2) = 4$, and the units ± 1 are incongruent (mod 4) in R_K , so $[\Sigma_4 : \Sigma] = 2$. (See [20].) Since $X(P) \in \Sigma_4$, Proposition 6.1 implies that $\frac{\beta-2}{2\beta}$, and therefore β , lies in Σ_4 . But the formulas in Proposition 4.1 imply that $\beta_1 = (\beta + 2i)/(2\beta)$ also lies in Σ_4 , so $i \in \Sigma_4$. The field Σ does not contain $i = \sqrt{-1}$ because the prime 2 is not ramified in Σ . Hence, $[\Sigma(i) : \Sigma] = 2$ shows that $\Sigma_4 = \Sigma(i)$.

We set $\beta = r + is$, with $r, s \in \Sigma$ and use $\beta^2 = r^2 - s^2 + 2rsi \in \Sigma$ to conclude that $rs = 0$. Thus, either β or $i\beta$ lies in Σ . Since β has only been determined up to multiplication by a power of i , we may assume that $\beta \in \Sigma$.

Finally, we examine the X -coordinate of the following point Q of order 4 on $E_3(\beta)$:

$$X(Q) = -\frac{1}{2} \left(\frac{\alpha}{\alpha + \beta\zeta_8^7} \right) = -\frac{1}{2} \left(\frac{\zeta_8\alpha}{\zeta_8\alpha + \beta} \right).$$

Since $\beta \in \Sigma$ and $E_3(\beta)$ has complex multiplication by R_K we deduce from this expression that $\zeta_8\alpha \in \Sigma_4$. Then $i\alpha^2 \in \Sigma$ and $\zeta_8\alpha = u + iv$ with $u, v \in \Sigma$ implies that $uv = 0$; so either $\zeta_8\alpha \in \Sigma$ or $\zeta_8^3\alpha \in \Sigma$. Set $\alpha_1 = \zeta_8^j\alpha \in \Sigma$, $j = 1$ or 3 . Then $\alpha^4 = -\alpha_1^4$ so the equation

$$16\alpha_1^4 - 16\beta^4 = \alpha_1^4\beta^4$$

shows that $16X^4 - 16Y^4 = X^4Y^4$ has a solution in Σ . Further, (α, β) is a solution of Fer_4 in $\Sigma_4(\zeta_8) = \Sigma(\zeta_8)$.

Theorem 6.2. The numbers β and $\zeta_8^j \alpha$, for $j = 1$ or $j = 3$, lie in the Hilbert class field Σ . The point $(X, Y) = (\zeta_8^j \alpha, \beta)$ is a nontrivial solution of the equation

$$16X^4 = X^4Y^4 + 16Y^4$$

in the field Σ . Thus, $x^4 + y^4 = z^4$ has a solution in the Hilbert class field of $Q(\sqrt{-d})$ whenever the discriminant $-d \equiv 1 \pmod{8}$.

We will convert $(X, Y) = (\zeta_8^j \alpha, \beta)$ into a solution of the form (1.2) using the following lemma.

Lemma 6.3. If the prime ideal factorization of 2 in R_K is $(2) = \wp_2 \wp_2'$, where $\wp_2 = (2, w)$ with $w = (v + \sqrt{-d})/2$ (as in Section 3), then $(\alpha) = \wp_2'^2$ in $\Sigma(\zeta_8)$ and $(\beta) = (2)\wp_2' = \wp_2 \wp_2'^2$ in Σ .

Proof. The assertion about (α) follows from (2.7) and classical results, since by (2.7) the ideal generated by (α^{12}) is

$$(\alpha^{12}) = \left(\frac{\eta(w/4)}{\eta(w/2)} \frac{\eta(w/2)}{\eta(w)} \right)^{24} = \left(2^{12} \frac{\Delta(\wp_2^2)}{\Delta(\wp_2)} \right) \left(2^{12} \frac{\Delta(\wp_2)}{\Delta(1)} \right) = \wp_2'^{12} \wp_2'^{12} = \wp_2'^{24}.$$

See [19, Satz 10] or [32, p. 114]. For the second assertion, the equation $\alpha^4(\beta^4 - 16) = 16\beta^4$ implies that $\wp_2' | \beta$ and $16(\alpha^4 + \beta^4) = \alpha^4\beta^4$ implies that $\wp_2 | \beta$. Thus, $2 | \beta$, and

$$\left(\frac{\beta}{2} \right)^4 - 1 = \left(\frac{\beta}{\alpha} \right)^4 \quad (6.3)$$

implies that $\alpha | \beta$. The last equation implies further that $\beta/2$ and β/α have no prime divisor in common. In particular, β is not divisible by any prime divisor of an odd prime in \mathbf{Z} . Also, $\beta/2$ cannot be divisible by a prime divisor of \wp_2 because β/α is, and β/α cannot be divisible by a prime divisor of \wp_2' because $\beta/2$ is. Hence, $(\beta) = 2\wp_2'$ in Σ and $(\beta/2)^2$ is a unit times α . \square

We may re-write (6.3) in the form

$$\left(\frac{\beta}{2} \right)^4 + \left(\frac{\beta}{\zeta_8^j \alpha} \right)^4 = 1, \quad (6.4)$$

where $(\beta/2) = \wp_2'$ and $(\beta/(\zeta_8^j \alpha)) = \wp_2$. Thus our solution of the Fermat quartic comes from ideal generators $\pi = \beta/(\zeta_8^j \alpha)$ and $\xi = \beta/2$ of \wp_2 and \wp_2' in the Hilbert

class field. This proves the first assertion of Theorem 1.1 of the Introduction and shows that the solution (1.1) is one of an infinite family.

All the same arguments apply in case $-d = -f^2d_1$, where f is odd and $d_1 \equiv 7 \pmod{8}$ is square-free. In that case the field Σ is replaced by the ring class field Ω_f of conductor f over K , and we use Proposition 3.3 along with the following result from the theory of complex multiplication.

Proposition 6.4. Assume the elliptic curve E is defined over the field Ω_f and has complex multiplication by the order R_{-d} of K . If \mathfrak{m} is an integral ideal of R_{-d} with $(f, \mathfrak{m}) = 1$, then the X -coordinates of points in $E[\mathfrak{m}]$ generate the field $\Sigma_{\mathfrak{m}}\Omega_f$ over Ω_f , where $\Sigma_{\mathfrak{m}}$ is the ray class field of conductor \mathfrak{m} over K .

This is an easy consequence of Theorem 2 of [13]. For our argument we take $\mathfrak{m} = (4) = 4R_{-d}$. Then the condition $(4, f) = 1$ implies by this proposition that the X -coordinates of points of either $E_1(\alpha)[4]$ or $E_3(\beta)[4]$ generate the field $\Sigma_4\Omega_f = \Omega_f(i)$ over Ω_f . This field is the same as Ω_{4f} , since a degree argument shows that $\Sigma_4 = \Omega_4$ and $\Omega_4\Omega_f = \Omega_{4f}$. The above arguments show that β and $\zeta_8^j\alpha$ lie in Ω_f for $j = 1$ or 3 , and lead to the following theorem.

Theorem 6.5. Let $-d = -f^2d_1$, with odd f and square-free $d_1 \equiv 7 \pmod{8}$. Further, let $(2) = \wp_2\wp'_2$ be the factorization of 2 into conjugate prime ideals $\wp_2 = (2, (v + \sqrt{-d})/2)$ and $\wp'_2 = (2, (v - \sqrt{-d})/2)$ in R_K , corresponding to the factorization of the ideal (2) in the order R_{-d} of discriminant $-d$ and conductor f in K . In the ring class field Ω_f of K the generators $\pi_f = \beta/(\zeta_8^j\alpha)$ and $\xi_f = \beta/2$ of the principal ideals $\wp_2R_{\Omega_f}$ and $\wp'_2R_{\Omega_f}$ satisfy

$$\pi_f^4 + \xi_f^4 = 1. \quad (6.5)$$

Each of π_f and ξ_f generates the ring class field Ω_f over K . \square

The last assertion of this theorem follows from (6.2), since $j(w/4)$ generates Ω_f over K . We will show in Section 8 that each of the numbers π_f and ξ_f generates Ω_f over \mathbb{Q} . This theorem shows that we have a lattice of solutions to the quartic Fermat equation over K , since the numbers π_f satisfy $\pi_f|\pi_{f'}$ in $\Omega_{f'}$ if $f|f'$. In fact, $\pi_{f'}/\pi_f$ is a unit in $\Omega_{f'}$.

For later use we require the following more general form of Lemma 6.3, whose proof carries over word for word. (See [10, pp. 32-33].)

Lemma 6.6. Assume $d = f^2d_1$, where f is odd and $d_1 \equiv 7 \pmod{8}$ is square-free. If $(2) = \wp_2\wp'_2$ in R_K , where $\wp_2 = (2, w)$ with $w = (v + \sqrt{-d})/2$ (as in Section 3), then $(\alpha) = \wp_2'^2$ in $\Omega_f(\zeta_8)$ and $(\beta) = (2)\wp_2' = \wp_2\wp_2'^2$ in Ω_f .

7 Application to the Legendre normal form.

We now apply the results of Section 6 to the λ -parameters of the Legendre normal form

$$E_\lambda : Y^2 = X(X - 1)(X - \lambda),$$

for which the curve E_λ has complex multiplication by the order R_{-d} of K , where $-d = -f^2 d_1 \equiv 1 \pmod{8}$. Values of λ satisfying this requirement may be found as roots of the equation

$$2^8(\lambda^2 - \lambda + 1)^3 - j(w/2)(\lambda^2 - \lambda)^2 = 0, \quad (7.1)$$

since $j(w/2)$ is a root of the class equation $H_{-d}(x) = 0$ (see (2.8) and the first paragraph of Section 3). Moreover, any λ for which E_λ has complex multiplication by R_{-d} is a conjugate of a root of (7.1) over K . Using (2.8) it is easy to see that $\lambda_1 = \alpha^4/16$ is one of the roots of this equation. It follows that the other roots are related to $\alpha^4/16$ by the anharmonic group.

(i) We note that $\lambda_1 = -\left(\frac{\zeta_8^j \alpha}{2}\right)^4$ for $j = 1$ or 3 . Thus $-\lambda_1$ and $-1/\lambda_1$ are fourth powers in the field Ω_f .

(ii) The value $\lambda_2 = 1 - \lambda_1 = 1 - \frac{\alpha^4}{16} = \frac{16 - \alpha^4}{16} = -\frac{\alpha^4}{\beta^4} = \left(\frac{\zeta_8^j \alpha}{\beta}\right)^4$, so λ_2 and $1/\lambda_2$ are fourth powers in Ω_f .

(iii) The value $\lambda_3 = 1 - \frac{1}{\lambda_1} = \frac{\alpha^4 - 16}{\alpha^4} = \left(\frac{2}{\beta}\right)^4$, so λ_3 and $1/\lambda_3$ are fourth powers in Ω_f .

Now we use Lemma 6.6. Since $(\lambda_1) = (\wp'_2/\wp_2)^4$, it is clear that neither λ_1 nor $1/\lambda_1$ are algebraic integers, while $1/\lambda_2$ and $1/\lambda_3$ are algebraic integers. Thus we have:

Theorem 7.1. Let λ be a complex number for which the Legendre normal form E_λ has complex multiplication by the order R_{-d} of K , where $-d = -f^2 d_1 \equiv 1 \pmod{8}$. Then λ lies in the ring class field Ω_f of K with conductor f . If either λ or $1/\lambda$ is an algebraic integer, then λ is a fourth power in the field Ω_f . If neither λ nor $1/\lambda$ is an algebraic integer, then $-\lambda$ is a fourth power in Ω_f .

The above computations and Lemma 6.6 also show that the only prime ideal divisors of Ω_f dividing the numerator or denominator of λ or $\lambda - 1$ in this theorem are prime divisors of 2.

8 Further properties of the solution.

We investigate further properties of the numbers α and β by making the substitution $\alpha^4 = (16\beta^4)/(\beta^4 - 16)$ in the j -invariant

$$j(w) = \frac{(\alpha^8 - 16\alpha^4 + 16)^3}{\alpha^8 - 16\alpha^4}$$

from (2.8). This substitution gives that

$$j(w) = \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4}. \quad (8.0)$$

Thus, β is a root of the polynomial

$$F_d(x) = (x^4(x^4 - 16)^4)^{h(-d)} H_{-d} \left(\frac{(x^8 + 224x^4 + 256)^3}{x^4(x^4 - 16)^4} \right). \quad (8.1)$$

Let $B(x) = B_d(x)$ denote the minimal polynomial of β over \mathbb{Q} . We now prove:

Proposition 8.1. The degree of β over \mathbb{Q} is $2h(-d)$, so that $\Omega_f = \mathbb{Q}(\beta)$. The numbers β and $-\beta$ are not conjugate over \mathbb{Q} , and β^4 has degree $2h(-d)$ over \mathbb{Q} .

Proof. Note that $\mathbb{Q}(\beta)$ contains $\mathbb{Q}(j(w))$ and is contained in Ω_f , so β has degree n over \mathbb{Q} satisfying $h(-d) \leq n \leq 2h(-d)$. If $n = h(-d)$, then β has to be real, by the choice of $w = (v + \sqrt{-d})/2$. This implies in turn that $j(w/4)$ is real, by (6.2). However, $j(w) = j(\mathbf{R}_{-d})$ is the j -invariant associated to the ideal class 1 in the ring class group (mod f) of K , while $j(w/4) = j(\wp_{2,-d}^2)$, where $\wp_{2,-d} = \wp_2 \cap \mathbf{R}_{-d}$. Since $\overline{j(\mathfrak{a})} = j(\mathfrak{a}^{-1})$ for any proper ideal \mathfrak{a} in \mathbf{R}_{-d} , $j(w/4)$ can be real if and only if \wp_2^2 has order 1 or 2 in the ring class group (mod f), so that the order of \wp_2 must divide 4. Now $\wp_2^4 \sim 1$ implies that $4 \cdot 2^4 = x^2 + dy^2$ with $x \equiv y \pmod{2}$ and $x \equiv y \equiv 2 \pmod{4}$ if x and y are even. This equation has a solution if and only if $d = 7, 15, 39, 55$ or 63 . We have the following class equations for these discriminants (see [14, vol. III] and [36]):

$$\begin{aligned}
H_{-7}(x) &= x + 3375, \\
H_{-15}(x) &= x^2 + 191025x - 121287375, \\
H_{-39}(x) &= x^4 + 331531596x^3 - 429878960946x^2 + 109873509788637459x \\
&\quad + 20919104368024767633, \\
H_{-55}(x) &= x^4 + 13136684625x^3 - 20948398473375x^2 + 172576736359017890625x \\
&\quad - 18577989025032784359375, \\
H_{-63}(x) &= x^4 + 67515199875x^3 - 193068841781250x^2 + 4558451243295023437500x \\
&\quad - 6256903954262253662109375.
\end{aligned}$$

Factoring the polynomial

$$x^{16h(-d)}(16 - x^4)^{h(-d)} H_{-d} \left(\frac{(x^8 - 256x^4 + 4096)^3}{x^{16}(16 - x^4)} \right)$$

(see (6.2)) in each of these cases shows that

$$\begin{aligned}
B_7(x) &= x^2 + 2x + 8 \text{ or } x^2 - 2x + 8; \\
B_{15}(x) &= x^4 + 8x^3 + 20x^2 + 16x + 64 \text{ or } x^4 - 8x^3 + 20x^2 - 16x + 64; \\
B_{39}(x) &= x^8 + 12x^7 + 168x^6 + 480x^5 + 848x^4 + 1728x^3 + 1536x^2 + 4096 \\
&\quad \text{or } x^8 - 12x^7 + 168x^6 - 480x^5 + 848x^4 - 1728x^3 + 1536x^2 + 4096; \\
B_{55}(x) &= x^8 + 12x^7 + 312x^6 - 672x^5 + 848x^4 - 2112x^3 - 768x^2 + 3072x + 4096 \\
&\quad \text{or } x^8 - 12x^7 + 312x^6 + 672x^5 + 848x^4 + 2112x^3 - 768x^2 - 3072x + 4096. \\
B_{63}(x) &= x^8 + 40x^7 + 440x^6 - 800x^5 + 784x^4 - 2560x^3 - 2560x^2 + 5120x + 4096 \\
&\quad \text{or } x^8 - 40x^7 + 440x^6 + 800x^5 + 784x^4 + 2560x^3 - 2560x^2 - 5120x + 4096.
\end{aligned}$$

Hence, the degree of β over \mathbb{Q} is $2h(-d)$ in all cases.

If β were conjugate to $-\beta$ over \mathbb{Q} , then β^2 would have degree $h(-d)$ over \mathbb{Q} , and since $j(w)$ and $j(w/4)$ are rational expressions in β^2 , the same argument as above would show that β^2 is real and $d = 7, 15, 39, 55$ or 63 . The above polynomials $B_d(x)$ show in these cases that β and $-\beta$ have different minimal polynomials. A similar argument shows that β^4 cannot have degree $h(-d)$ over \mathbb{Q} , so its degree must be $2h(-d)$. This proves the proposition. \square

As in [27, p. 256] we set $r(x) = \frac{(x^8 + 224x^4 + 256)^3}{x^4(x^4 - 16)^4}$ and use the fact that the algebraic extension $\mathbb{Q}(i, x)/\mathbb{Q}(i, r(x))$ is normal with a Galois group \hat{G}_{24} isomorphic

to the octahedral group. Since β generates the field Ω_f over \mathbb{Q} and $\Omega_f \cap \mathbb{Q}(i) = \mathbb{Q}$ (2 is unramified in Ω_f), it is clear that $B(x)$ is irreducible over $\mathbb{Q}(i)$. Now the group \hat{G}_{24} is generated by the substitutions

$$x \rightarrow ix, \quad x \rightarrow \frac{2x+4}{x-2}.$$

Since the polynomial $H_{-d}(z)$ is irreducible over $\mathbb{Q}(i)$, it corresponds to a prime divisor \mathfrak{p} of the field $\mathbb{Q}(i, z)$. The irreducible factors of $F_d(x)$ over $k = \mathbb{Q}(i)$ correspond to the prime divisors of the field $k(x), z = r(x)$, which extend \mathfrak{p} . As noted above, the extension $k(x)/k(z)$ is finite and normal with Galois group \hat{G}_{24} . Hence, \hat{G}_{24} is transitive on the irreducible factors of $F_d(x)$ over k . Since these factors have degree $2h(-d)$ over k , by Proposition 8.1, the orbit of $B(x)$ has 12 elements, and the stabilizer of $B(x)$ has order 2.

We now let S denote the subgroup of \hat{G}_{24} consisting of the linear fractional mappings

$$x \rightarrow \pm x, \quad \pm \frac{4}{x}, \quad \pm \frac{2(x+2)}{x-2}, \quad \pm \frac{2(x-2)}{x+2}. \quad (8.2)$$

Note that S is a 2-Sylow subgroup of \hat{G}_{24} and that it is isomorphic to the dihedral group of order 8. It consists of all the linear fractional maps in \hat{G}_{24} which don't involve $i = \sqrt{-1}$ as a coefficient.

Proposition 8.2. The stabilizer of $B(x) = B_d(x)$ in \hat{G}_{24} is one of the two subgroups

$$S_1 = \left\{ (x \rightarrow x), \left(x \rightarrow \frac{2(x+2)}{x-2} \right) \right\} \quad \text{or} \quad S_2 = \left\{ (x \rightarrow x), \left(x \rightarrow \frac{-2(x-2)}{x+2} \right) \right\}. \quad (8.3)$$

One of these subgroups is the stabilizer of $B_d(x)$ and the other is the stabilizer of $B_d(-x)$.

Proof. The polynomial $B(x)$ is a normal polynomial, since its root β generates the normal extension Ω_f/\mathbb{Q} . The stabilizer of $B(x)$ must therefore be a subgroup of S , because the other elements of \hat{G}_{24} map the roots of $B(x)$ to numbers which do not lie in Ω_f . For example, the mapping

$$x \rightarrow \frac{-2i(ix+2)}{ix-2} = \frac{-2i(x-2i)}{x+2i}$$

cannot map β to an element of Ω_f , because otherwise

$$-2i\beta - 4 = \beta\gamma + 2i\gamma, \quad \text{with } \gamma \in \Omega_f,$$

implies $\gamma = -\beta$ and $\beta = \pm 2 \in \mathbb{Q}$, which is not the case. Now the maps $x \rightarrow \frac{-2(x+2)}{x-2}$ and $x \rightarrow \frac{2(x-2)}{x+2}$ cannot fix $B(x)$ because they have order 4 and the stabilizer has order 2. Furthermore, the maps $x \rightarrow \pm 4/x$ cannot fix $B(x)$ because the principal ideal generated by β in the ring of integers of Ω_f is $2\wp'_2 = \wp_2\wp_2'^2$ and the principal ideal generated by $4/\beta$ is \wp_2 (by Lemma 6.6). Hence β and $\pm 4/\beta$ are divisible by different numbers of prime ideals in Ω_f and therefore cannot be conjugate over \mathbb{Q} . Finally, Proposition 8.1 shows that β and $-\beta$ are not conjugate over \mathbb{Q} , so only two nontrivial mappings remain in (8.2) as possibilities. If one of these lies in the stabilizer of $B(x)$, then the other lies in the stabilizer of $B(-x)$. This proves the proposition. \square

Corollary. Assume that the stabilizer of $B_d(x)$ is the group S_1 in (8.3). The roots of the minimal polynomial $b_d(x) = B_d(2x)/2^{2h(-d)}$ of $\beta/2$ over \mathbb{Q} are mapped into themselves by the mapping $x \rightarrow \frac{x+1}{x-1}$.

It follows from this corollary and $(\beta/2) = \wp'_2$ that $((\beta+2)/(\beta-2)) = \wp_2$, because $\beta/2+1$ and $\beta/2-1$ are relatively prime to \wp'_2 . This gives the following result.

Proposition 8.3. Choose the sign of β so that the stabilizer of $B_d(x)$ is the group S_1 in (8.3). Then the number $\gamma = \frac{\beta(\beta+2)}{4(\beta-2)}$ is a unit in Ω_f with degree $h(-d)$ over \mathbb{Q} , and $\mathbb{Q}(\gamma) = \mathbb{Q}(j(w))$.

Proof. The fact that γ is a unit follows from the ideal factorization

$$\left(\frac{\beta}{2}\right) \left(\frac{\beta+2}{\beta-2}\right) = \wp'_2 \wp_2 = (2).$$

Since the map $\sigma : \beta/2 \rightarrow (\beta+2)/(\beta-2)$ has order 2, it is clear that γ is invariant under σ . Its degree over \mathbb{Q} is therefore at most $h(-d)$. If γ had degree $k < h(-d)$ over \mathbb{Q} , then β would satisfy an equation of degree $2k$ over \mathbb{Q} , which contradicts

Proposition 8.1. Moreover, the automorphism $\beta \rightarrow 2(\beta + 2)/(\beta - 2)$ fixes $j(w)$ by (8.0), which implies $\mathbb{Q}(j(w)) = \mathbb{Q}(\gamma)$, since $j(w)$ and γ have the same degree over \mathbb{Q} . \square

The units in Proposition 8.3 are similar to the units discussed in [36], since they have degree $h(-d)$ over \mathbb{Q} and generate Ω_f over K , and their minimal polynomials have small coefficients (though generally not as small as the coefficients of the polynomials $W_{-d}(x)$ in [36]). See Theorem 8.6 below and the examples in Section 12.

Note that the rational function $r(x)$ can be expressed as:

$$r(x) = \frac{(x^8 + 224x^4 + 256)^3}{x^4(x^4 - 16)^4} = \frac{(4z^2 + 1)^3(4z^2 - 8z + 1)^3}{z^4(2z - 1)^4}, \quad z = \frac{x(x + 2)}{4(x - 2)},$$

so we have the formula

$$j(w) = \frac{(4\gamma^2 + 1)^3(4\gamma^2 - 8\gamma + 1)^3}{\gamma^4(2\gamma - 1)^4}.$$

Proposition 8.4. The numbers $\beta/2$ and $\pm\beta/(\zeta_8^j\alpha)$ are conjugate over \mathbb{Q} .

Proof. From the arguments which precede Theorem 7.1 we know that the numbers

$$\lambda_1 = -\left(\frac{\zeta_8^j\alpha}{2}\right)^4, \quad \frac{1}{\lambda_2} = \frac{1}{1 - \lambda_1} = \left(\frac{\beta}{\zeta_8^j\alpha}\right)^4, \quad \frac{1}{\lambda_3} = \left(\frac{\beta}{2}\right)^4 \quad (8.4)$$

are roots of the polynomial

$$L_d(x) = (x^2 - x)^{2h(-d)} H_{-d} \left(\frac{2^8(x^2 - x + 1)^3}{(x^2 - x)^2} \right).$$

Furthermore, the numbers in (8.4) have degree $2h(-d)$ over \mathbb{Q} , by Proposition 8.1 and the fact that α^4 and β^4 generate the same extension of \mathbb{Q} . From Lemma 6.6 it is clear that the minimal polynomials of $\lambda_1, 1/\lambda_3$, and λ_3 over \mathbb{Q} are all distinct, because these numbers do not have conjugate prime ideal factorizations:

$$(\lambda_1) = \frac{\wp_2'^4}{\wp_2^4}, \quad \left(\frac{1}{\lambda_3}\right) = \wp_2'^4, \quad (\lambda_3) = \frac{1}{\wp_2'^4}.$$

Denoting the respective minimal polynomials by $g_i(x)$ we have the factorization $L_d(x) = g_1(x)g_2(x)g_3(x)$ since the degrees match. The roots of $g_2(x)$ are the roots of $L_d(x)$ which are algebraic integers; the roots of $g_3(x)$ are the roots of $L_d(x)$ whose reciprocals are algebraic integers; and the roots of $g_1(x)$ are the remaining roots of $L_d(x)$. Since $1/\lambda_2$ and $1/\lambda_3$ are algebraic integers, they are both roots of the polynomial $g_2(x)$. Hence, $\left(\frac{\beta}{\zeta_8^j \alpha}\right)^4$ and $\left(\frac{\beta}{2}\right)^4$ are conjugate over \mathbb{Q} . It follows that $\beta/2$ and $\pm\beta/(\zeta_8^j \alpha)$ are conjugate over \mathbb{Q} . \square

Remarks. 1. Having chosen the sign of β so that the corollary to Proposition 8.2 holds, we may replace the root of unity ζ_8^j by ζ_8^{j+4} , if necessary, to guarantee that $\xi = \beta/2$ and $\pi = \beta/(\zeta_8^j \alpha)$ are conjugates, where $j \in \{1, 3, 5, 7\}$. The residue class of $j \pmod{8}$ is uniquely determined by this requirement.

2. It is not hard to see that $\pm\pi \pm \xi \neq 1$. If, for example, $\pi + \xi = 1$, then $\pi = 1 - \xi$ and ξ are conjugates, so the minimal polynomial $b_d(x)$ over \mathbb{Q} of ξ satisfies $b_d(1 - x) = b_d(x)$. Then the composition of the maps $\sigma = (x \rightarrow 1 - x)$ and $\tau = (x \rightarrow (x + 1)/(x - 1))$ would leave the roots of $b_d(x)$ invariant. But this composition is $\sigma\tau = (x \rightarrow -2/(x - 1))$, which has infinite order in the group of Möbius transformations. This is not possible if $d > 7$, since the only fixed points of powers of $\sigma\tau$ have degree 2 over \mathbb{Q} . A similar argument applies to show that $\pm\pi \pm \xi \neq 1$. This remark proves the second assertion of Theorem 1.1. It can be shown in the same way that the point (π, ξ) does not lie on *any* rational line.

Proposition 8.5. Choose the sign of β and the root of unity ζ_8^j so that $\frac{\beta + 2}{\beta - 2}$ and $\pi = \frac{\beta}{\zeta_8^j \alpha}$ are conjugate to $\xi = \beta/2$ over \mathbb{Q} . If $\tau = \left(\frac{\Omega_f/K}{\wp_2}\right)$ is the automorphism associated to the ideal \wp_2 by the Artin map for Ω_f/K , then

$$\pi = \frac{\beta^{\tau^{-2}} + 2}{\beta^{\tau^{-2}} - 2} = \frac{\xi^{\tau^{-2}} + 1}{\xi^{\tau^{-2}} - 1}.$$

Equivalently,

$$\xi = \frac{\pi^{\tau^2} + 1}{\pi^{\tau^2} - 1}.$$

Remark. This proposition verifies the last assertion of Theorem 1.1.

Proof. Let ψ be the automorphism in $\Gamma = \text{Gal}(\Omega_f/\mathbb{Q})$ for which

$$\xi^\psi = \left(\frac{\beta}{2}\right)^\psi = \pi = \frac{\beta}{\zeta_8^j \alpha}$$

and $\sigma \in \Gamma$ the automorphism for which

$$\xi^\sigma = \left(\frac{\beta}{2}\right)^\sigma = \frac{\beta + 2}{\beta - 2}.$$

These are both well-defined automorphisms of $\Omega_f = \mathbb{Q}(\beta)$ over \mathbb{Q} and they both have order 2 since they interchange \wp_2 and \wp'_2 . Furthermore, we have

$$(\beta^4)^\psi = -16 \frac{\beta^4}{\alpha^4} = 16 - \beta^4$$

and

$$\beta^\sigma = \frac{2(\beta + 2)}{\beta - 2}.$$

By (8.2), (8.0), and (6.2) we have

$$\begin{aligned} j(w)^{\sigma\psi} &= \left(\frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4} \right)^{\sigma\psi} = \left(\frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4} \right)^\psi \\ &= \frac{(\beta^8 - 256\beta^4 + 4096)^3}{\beta^{16}(16 - \beta^4)} = j\left(\frac{w}{4}\right). \end{aligned}$$

Hence, $\sigma\psi$ is an automorphism in $\text{Gal}(\Omega_f/K)$ taking $j(w)$ to $j(w/4)$. On the other hand,

$$j\left(\frac{w}{4}\right)^{\tau^2} = j(\wp_{2,-d}^2)^{\tau^2} \equiv j(\wp_{2,-d}^2)^4 \equiv j(\wp_{2,-d}^{-2} \wp_{2,-d}^2) = j(w) \pmod{\wp_2},$$

by Hasse's congruence [19, Satz 11], [10, p.34]; and therefore $j\left(\frac{w}{4}\right)^{\tau^2} = j(w)$, since $p = 2$ does not divide the discriminant of $H_{-d}(x)$ (see [8]). It follows that $\sigma\psi = \tau^{-2}$ and therefore $\psi = \sigma\tau^{-2}$. This implies the assertion of the proposition. \square

Theorem 8.6. If the sign of β is chosen as in Proposition 8.3, so that β and $\frac{2(\beta + 2)}{\beta - 2}$ are conjugates over \mathbb{Q} , then:

- a) the quantities $\beta^3(\beta^2 - 4)$ and $-\beta(\beta^2 + 4)$ are fourth powers in Ω_f ;
- b) the unit $\gamma = \frac{\beta(\beta + 2)}{4(\beta - 2)}$ is a square in $\mathbb{Q}(\gamma)$; and
- c) the quantity $\beta(\beta - 2)/8$ is a unit, with $N_{\mathbb{Q}(\gamma)}(\beta(\beta - 2)/8) = \gamma$.
- d) $2\gamma - 1 = \frac{\beta^2 + 4}{2(\beta - 2)}$ is a unit.

Proof. From Proposition 8.5 we have that

$$\pi = \frac{\beta}{\alpha_1} = \frac{\beta^{\tau^{-2}} + 2}{\beta^{\tau^{-2}} - 2},$$

so Theorem 1.1 implies the equation

$$(\beta^{\tau^{-2}} + 2)^4 + \left(\frac{\beta}{2}\right)^4 (\beta^{\tau^{-2}} - 2)^4 = (\beta^{\tau^{-2}} - 2)^4.$$

From this it follows that

$$\left(\frac{\beta}{2}\right)^4 (\beta^{\tau^{-2}} - 2)^4 = (\beta^{\tau^{-2}} - 2)^4 - (\beta^{\tau^{-2}} + 2)^4 = -16\beta^{\tau^{-2}}(\beta^{2\tau^{-2}} + 4).$$

Applying the automorphism τ^2 to both sides gives

$$\left(\frac{\beta^{\tau^2}}{2} \left(\frac{\beta}{2} - 1\right)\right)^4 = -\beta(\beta^2 + 4). \quad (8.5)$$

Now we apply the automorphism σ taking β to $\frac{2(\beta + 2)}{\beta - 2}$:

$$\left(\frac{\beta^{\tau^2\sigma}}{2} \left(\frac{\beta^\sigma}{2} - 1\right)\right)^4 = -\beta^\sigma((\beta^\sigma)^2 + 4) = \frac{-16(\beta + 2)(\beta^2 + 4)}{(\beta - 2)^3}. \quad (8.6)$$

The right side of this equation is equal to

$$\frac{-16(\beta + 2)(\beta^2 + 4)}{(\beta - 2)^3} = \frac{16\beta^3(\beta^2 - 4) \times [-\beta(\beta^2 + 4)]}{\beta^4(\beta - 2)^4}. \quad (8.7)$$

Equations (8.5) and (8.6) now clearly imply that $\beta^3(\beta^2 - 4)$ is a fourth power in Ω_f , as claimed. This proves part a).

It is immediate that γ is a square in Ω_f . To prove part b) we have to show that $\sqrt{\gamma}$ is invariant under the automorphism σ . From (8.5)-(8.7) we have

$$\sqrt{\gamma} = \frac{\beta}{8} \frac{\beta^{2\tau^2\sigma}(\beta^\sigma - 2)^2}{\beta^{2\tau^2}(\beta - 2)},$$

so

$$\sqrt{\gamma}^\sigma = \frac{\beta^\sigma}{8} \frac{\beta^{2\tau^2}(\beta - 2)^2}{\beta^{2\tau^2\sigma}(\beta^\sigma - 2)} = \frac{1}{\sqrt{\gamma}} \frac{\beta\beta^\sigma}{64} (\beta - 2)(\beta^\sigma - 2) = \frac{1}{\sqrt{\gamma}} \frac{\beta(\beta + 2)}{4(\beta - 2)} = \sqrt{\gamma}.$$

This proves the assertion.

The last equation makes it clear that γ is the norm of the quantity $\frac{\beta(\beta - 2)}{8}$ in the extension $\Omega_f/\mathbb{Q}(\gamma)$. To see that this number is also a unit, recall from Lemma 6.6 that $(\beta) = \wp_2 \wp_2'^2$, so that $(\beta - 2) = \wp_2 \wp_2' \mathfrak{a}$, where $(\mathfrak{a}, \wp_2') = 1$. Now note that

$$(\beta^\sigma - 2)(\beta - 2) = \left(\frac{2\beta + 4}{\beta - 2} - 2 \right) (\beta - 2) = 8.$$

Thus, the norm to $L = \mathbb{Q}(\gamma)$ of $(\beta/2 - 1) = \mathfrak{a}$ is $(2) = \wp_2 \wp_2'$. If a prime ideal \mathfrak{p} of R_{Ω_f} divides \mathfrak{a} , then its conjugate \mathfrak{p}^σ divides \wp_2' . But \mathfrak{p}^2 cannot divide \mathfrak{a} because $N_{\Sigma/L}(\mathfrak{a}) = (2)$ is square-free in L . Now $\mathfrak{a}\mathfrak{a}^\sigma = \wp_2 \wp_2'$ and $\mathfrak{a}|\wp_2$ implies that $\mathfrak{a} = \wp_2$. Hence, $(\beta - 2) = \wp_2^2 \wp_2'$, and this implies that $\frac{\beta(\beta - 2)}{8}$ is a unit. This proves part c).

To prove d), first use $(\beta - 2) = \wp_2^2 \wp_2'$, $(\beta + 2) = (4(\beta - 2)/\beta)$, and Lemma 6.6 to deduce that $(\beta + 2) = \wp_2^3 \wp_2'$. Then use $\beta^4 - 16 = 16\beta^4/\alpha^4$ to see that $(\beta^4 - 16) = \wp_2^8 \wp_2'^4$. Putting these facts together gives that $(\beta^2 + 4) = \wp_2^3 \wp_2'^2 = (2(\beta - 2))$. This proves the assertion. \square

Corollary 1. The point $(\beta, 2(\beta - 2)\sqrt{\gamma})$ is an integral point on the elliptic curve $E : Y^2 = X(X^2 - 4)$ with coordinates in the ring class field Ω_f of K with conductor f .

Corollary 2. The point $\left(-\beta, \frac{1}{16}\beta^{2\tau^2}(\beta - 2)^2\right)$ is an integral point on the elliptic curve $\bar{E} : Y^2 = X(X^2 + 4)$ with coordinates in the ring class field Ω_f .

Note that the curves in Corollary 1 and 2 are curves (64B) and (32B), respectively, in Cremona's tables [6].

Corollary 3. The algebraic integer $-\beta$ is the sum of two squares in the ring R_{Ω_f} . Moreover, β is a sum of two squares in Ω_f if and only if the order of $\wp_{2,-d} = \wp_2 \cap \mathbf{R}_{-d}$ in the class group of \mathbf{R}_{-d} is even, or equivalently, iff the order of \wp_2 in the ring class group (mod f) is even.

Proof. By part a) of the theorem, we have

$$-8\beta^3 = \beta^3(\beta^2 - 4) - \beta^3(\beta^2 + 4) = A^4 + \beta^2 B^4,$$

for some integers A, B in Ω_f . Thus,

$$-\beta = \left(\frac{A^2 + \beta B^2}{4\beta} \right)^2 + \left(\frac{-A^2 + \beta B^2}{4\beta} \right)^2,$$

where the ideals $(A)^4 = \wp_2^8 \wp_2'^8$ and $(B)^4 = \wp_2^4 \wp_2'^4$ by the computations in the above proof; hence the expressions inside the squares are algebraic integers. It follows that β is a sum of two squares in Ω_f if and only if -1 is a sum of two squares, which holds if and only if the level (*Stufe*) of Ω_f is 2. The characterization of the level $s(\Omega_f)$ in terms of the order of $\wp_{2,-d}$ in $\text{Pic}(\mathbf{R}_{-d})$ follows from [18, Satz 14] (see also [12]), since the absolute degree of the prime divisors of 2 in Ω_f is equal to the order of \wp_2 in the ring class group (mod f) of K . \square

Corollary 4. In the ring R_{Ω_f} the algebraic integer $j(w)$ is a cube times the fourth power of a unit.

Proof. This follows from Part d) of Theorem 8.6 and the formula

$$j(w) = \frac{(4\gamma^2 + 1)^3(4\gamma^2 - 8\gamma + 1)^3}{\gamma^4(2\gamma - 1)^4}.$$

\square

Remark. We also note the formula

$$j(w) = \frac{(2\gamma + i)^3(2\gamma - i)^3(2\gamma - 2 + \sqrt{3})^3(2\gamma - 2 - \sqrt{3})^3}{\gamma^4(2\gamma - 1)^4},$$

which expresses $j(w)$ as γ^{-4} times a product of powers of quantities of the form $(2\gamma - \varepsilon)$, where ε is a unit.

To prove that $\alpha_1 = \zeta_8^j \alpha$ is a square in Ω_f we start with the following.

Theorem 8.7. If α is any algebraic number for which the curve $E_1(\alpha)$ (see (2.10)) has complex multiplication by the order R_{-d} in $K = \mathbb{Q}(\sqrt{-d})$, where $-d = -f^2 d_1 \equiv 1 \pmod{8}$, then α lies in the class field $\Sigma_{16}\Omega_f$ over K , where Σ_{16} is the ray class field of conductor 16 over K .

Proof. Assume that $E_1 = E_1(\alpha)$ has complex multiplication by R_{-d} . From (8.0) and (8.1) and the discussion preceding (8.2), it is clear that any algebraic number β for which (α, β) lies on Fer_4 must lie in the field $\Omega_f(i)$. This is because β is then a root of the polynomial $F_d(x)$ in (8.1), and its minimal polynomial is in the orbit of the polynomial $B(x) = B_d(x)$ under the action of the group \hat{G}_{24} . Hence, $b = 1/\alpha^4 = 1/16 - 1/\beta^4$ lies in $\Omega_f(i)$ also, and therefore E_1 is defined over this field. It follows from Proposition 6.4 (see [13]) that the X -coordinates of points of order 16 on E_1 lie in $\Sigma_{16}\Omega_f$, since $i \in \Sigma_4 \subset \Sigma_{16}$.

Now we use the fact that there is a cyclic isogeny $\rho : E_1 \rightarrow E_3(\alpha)$, where $\rho = \phi \circ \psi$ is given by the equations immediately preceding equation (2.12). Let Q be the point on $E_3 = E_3(\alpha)$ whose X -coordinate is

$$\xi = -\frac{1}{2}\gamma_1 = -\frac{1}{2}\frac{\beta}{\beta + \zeta_8\alpha},$$

from Proposition 5.1 (switching the roles of α and β). Let P be a point on E_1 for which $\rho(P) = Q$. Then $[4]Q = O$ on E_3 implies that $\rho([4]P) = O$ on E_3 , and therefore, since ρ has degree 4, that $P \in E_1[16]$. Since the coefficients of the rational function $\phi_1 \circ \psi_1$ depend only on b , they lie in the field $\Omega_f(i)$. Hence, $\xi = \phi_1 \circ \psi_1(X(P))$ shows that $\xi \in \Sigma_{16}\Omega_f$ and therefore $\zeta_8\alpha \in \Sigma_{16}\Omega_f$.

On the other hand, it is easy to see that $\zeta_8 \in \Sigma_8 \subset \Sigma_{16}$, by the transference theorem (*Verschiebungssatz*) of class field theory (see [20]). By that theorem, $K(\zeta_8)$ is class field over K for the ideal group H consisting of the ideals \mathfrak{a} of K (prime to 2) whose norms to \mathbb{Q} are $1 \pmod{8}$. However, any principal ideal in R_K with a generator congruent to $1 \pmod{8}$ satisfies this condition. This shows that the ideal group corresponding to Σ_8 lies in H , and therefore $K(\zeta_8) \subset \Sigma_8$. (A similar argument shows of course that $\zeta_{16} \in \Sigma_{16}$ also.)

It follows that $\alpha \in \Sigma_{16}\Omega_f$. \square

Computations on Maple suggest the following stronger result.

Conjecture. If α is an algebraic number for which the curve $E_1(\alpha)$ has complex multiplication by the order \mathbf{R}_{-d} in $K = \mathbb{Q}(\sqrt{-d})$, where $-d = -f^2 d_1 \equiv 1 \pmod{8}$, then α lies in the ring class field $\Omega_{16}\Omega_f = \Omega_{16f}$ over K .

Next, we need the following result.

Theorem 8.8. (See [26].) Let $j = j(\alpha) = \frac{(\alpha^8 - 16\alpha^4 + 16)^3}{\alpha^4(\alpha^4 - 16)}$ be the j -invariant of the curve $E_1(\alpha)$, where α is an *indeterminate*. Also, let k be any field whose characteristic is not 2 that contains a primitive 8-th root of unity $\zeta_8 = (1+i)/\sqrt{2}$. Then the normal closure of the algebraic extension $k(\alpha)/k(j)$ is the function field

$$N = k(\zeta_{16}\beta^{1/2}, \zeta_{16}\beta^{1/4}(\beta^2 - 4)^{1/4}, \beta^{1/4}(\beta^2 + 4)^{1/4}),$$

where (α, β) satisfies the equation $Fer_4 : 16\alpha^4 + 16\beta^4 = \alpha^4\beta^4$. Moreover, $k(\alpha, \beta) \subset N$.

Proof. Let $F(x, j)$ denote the polynomial

$$F(x, j) = (x^2 - 16x + 16)^3 - jx(x - 16).$$

We are looking for the splitting field N of $F(x^4, j) = 0$ over $k(j)$. Note first that

$$F(8 \pm \sqrt{y + 48}, j) = y^3 - jy + 16j = G(y, j).$$

From (8.0) we have

$$j = \frac{(\alpha^8 - 16\alpha^4 + 16)^3}{\alpha^4(\alpha^4 - 16)} = \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4}.$$

With this substitution for j in $G(y, j)$, the latter polynomial factors over $k(\beta)$ as follows:

$$\begin{aligned} G(y, j) &= \frac{1}{\beta^4(\beta^4 - 16)^4} ((\beta^6 - 8\beta^4 + 16\beta^2)y + \beta^8 + 224\beta^4 + 256) \\ &\quad \times ((\beta^8 - 32\beta^4 + 256)y - 16\beta^8 - 3584\beta^4 - 4096) \\ &\quad \times ((\beta^6 + 8\beta^4 + 16\beta^2)y - \beta^8 - 224\beta^4 - 256). \end{aligned}$$

Denote the respective roots of $G(y, j) = 0$ by

$$\xi_1 = -\frac{\beta^8 + 224\beta^4 + 256}{\beta^2(\beta^2 - 4)^2},$$

$$\xi_2 = 16 \frac{\beta^8 + 224\beta^4 + 256}{(\beta^4 - 16)^2},$$

$$\xi_3 = \frac{\beta^8 + 224\beta^4 + 256}{\beta^2(\beta^2 + 4)^2}.$$

Now a straightforward calculation on Maple shows that the roots of $F(x, j) = 0$ are:

$$8 \pm \sqrt{\xi_1 + 48} = \frac{i(\beta - 2i)^4}{\beta(\beta^2 - 4)}, \quad \frac{-i(\beta + 2i)^4}{\beta(\beta^2 - 4)};$$

$$8 \pm \sqrt{\xi_2 + 48} = \frac{16\beta^4}{\beta^4 - 16}, \quad \frac{-256}{\beta^4 - 16};$$

$$8 \pm \sqrt{\xi_3 + 48} = \frac{(\beta + 2)^4}{\beta(\beta^2 + 4)}, \quad \frac{-(\beta - 2)^4}{\beta(\beta^2 + 4)}.$$

The third root listed above is $8 + \sqrt{\xi_2 + 48} = \alpha^4$. Noting that $\zeta_8 \in k$, this shows that the splitting field of $F(x^4, j) = 0$ is generated over $k(j)$ by the elements

$$\beta, \quad \zeta_{16}\beta^{1/4}(\beta^2 - 4)^{1/4}, \quad (\beta^4 - 16)^{1/4}, \quad \beta^{1/4}(\beta^2 + 4)^{1/4}.$$

Since

$$\frac{(\zeta_{16}\beta^{1/4}(\beta^2 - 4)^{1/4}) (\beta^{1/4}(\beta^2 + 4)^{1/4})}{(\beta^4 - 16)^{1/4}} = \zeta_{16}\beta^{1/2},$$

this proves the assertion. \square

Now take $k = \Sigma_{16}\Omega_f$ in Theorem 8.8. As noted in the proof of Theorem 8.7, this field satisfies the hypothesis of Theorem 8.8.

If j_0 is any root of the class equation $H_{-d}(x) = 0$, and P_{j_0} is the prime divisor of the function field $k(j)$ corresponding to the numerator divisor of $j - j_0$, then any α_0 for which $j(\alpha_0) = j_0$ lies in $k = \Sigma_{16}\Omega_f$, by Theorem 8.7; because the j -invariant of $E_1(\alpha_0)$ is equal to j_0 , this curve has complex multiplication by \mathbf{R}_{-d} . Hence the polynomial

$$F(x^4, j_0) = (x^8 - 16x^4 + 16)^3 - j_0x^4(x^4 - 16) = 0,$$

which is the specialization for $j = j_0$ of the minimal polynomial of α over $k(j)$, splits completely over k . Therefore, the prime divisor P_{j_0} splits into prime divisors of degree 1 in $k(\alpha)$, and it follows that P_{j_0} splits into prime divisors of degree 1 in the normal closure N of $k(\alpha)/k(j)$.

This has the following consequence. Let \bar{P} be any prime divisor of N lying over P_{j_0} . Then the residue class field of $N \bmod \bar{P}$ is just the field k . Hence, the residue mod \bar{P} (i.e., the specialization) of any element in N which is integral for \bar{P} lies in k . In particular, if (α_0, β_0) is a solution of Fer_4 for which $\zeta_8^j \alpha_0$ and β_0 lie in Ω_f , then because $k(\alpha, \beta) \subset N$ there is a prime divisor \bar{P} of N for which

$$\alpha \equiv \alpha_0, \quad \beta \equiv \beta_0 \pmod{\bar{P}},$$

and consequently the residue of $\zeta_{16} \beta^{1/2} \in N$ also lies in $k = \Sigma_{16} \Omega_f$. Switching back to our usual notation, we have:

Proposition 8.9. If (α, β) is a solution of Fer_4 for which $\zeta_8^j \alpha$ and β lie in Ω_f , then the quantity $\beta^{1/2} \in \Sigma_{16} \Omega_f$ generates an abelian extension of K .

We can now prove

Theorem 8.10. If β and ζ_8^j are chosen as in Proposition 8.5, then $\alpha_1 = \zeta_8^j \alpha$ is a square in Ω_f .

Proof. From Proposition 8.5, we know that

$$\frac{\beta}{\alpha_1} = \frac{\beta^{\tau^{-2}} + 2}{\beta^{\tau^{-2}} - 2},$$

so that

$$\alpha_1 = \frac{\beta(\beta^{\tau^{-2}} - 2)}{\beta^{\tau^{-2}} + 2}.$$

This equation implies further that

$$\alpha_1 = \frac{\beta}{\beta^{\tau^{-2}}} \frac{\beta^{\tau^{-2}}(\beta^{\tau^{-2}} - 2)(\beta^{\tau^{-2}} + 2)}{(\beta^{\tau^{-2}} + 2)^2}.$$

By Theorem 8.6, the quantity

$$\beta^{\tau^{-2}}(\beta^{\tau^{-2}} - 2)(\beta^{\tau^{-2}} + 2) = f(\beta)^{\tau^{-2}},$$

with $f(\beta) = \beta(\beta^2 - 4)$, is a square in Ω_f . Hence, $\alpha_1 = \zeta_8^j \alpha$ is a square in Ω_f if and only if the unit $\beta^{1-\tau^{-2}} = \frac{\beta}{\beta^{\tau^{-2}}}$ is a square in Ω_f .

Now we appeal to Proposition 8.9. By that result $L = \Omega_f(\beta^{1/2})$ is an abelian extension of K . There is an isomorphism of L over K which extends the automorphism

$\tau^{-2} \in \text{Gal}(\Omega_f/K)$ on Ω_f/K . Since L is normal over K , this is an automorphism, which we again denote by τ^{-2} . Hence,

$$L = L^{\tau^{-2}} = \Omega_f \left((\beta^{1/2})^{\tau^{-2}} \right) = \Omega_f \left((\beta^{\tau^{-2}})^{1/2} \right).$$

The field L is a quadratic extension of Ω_f , so by Kummer theory we have that

$$\beta = \beta^{\tau^{-2}} \eta^2, \quad \eta \in \Omega_f.$$

Hence, $\beta^{1-\tau^{-2}}$ is a square in Ω_f . \square

Corollary. If $-d = f^2 d_1 \equiv 1 \pmod{8}$ and w is chosen as in Proposition 3.3, then there is a unique value of $j \in \{1, 3, 5, 7\}$ for which

$$\zeta_8^{(j-1)/2} \frac{\eta(w/4)}{\eta(w)} \in \Omega_f.$$

Remark. Theorem 10.6 gives a formula for the value of j in this corollary, in the case that $w = (v + \sqrt{-d})/2$ and $v = 1$ or 3 .

We can derive one more interesting consequence from Theorem 8.6 and its Corollary 1. We start with the elementary observation that on the curve

$$E: Y^2 = X(X^2 - 4)$$

the sum of the points $P_d = (\beta, 2(\beta - 2)\sqrt{\gamma})$ and $(2, 0)$ is

$$P_d + (2, 0) = (\beta, 2(\beta - 2)\sqrt{\gamma}) + (2, 0) = \left(\frac{2(\beta + 2)}{\beta - 2}, \frac{-16\sqrt{\gamma}}{\beta - 2} \right) = -P_d^\sigma,$$

since $2(\beta^\sigma - 2) = 16/(\beta - 2)$ and $\sqrt{\gamma^\sigma} = \sqrt{\gamma}$, by the proof of Theorem 8.6b). This gives that

$$P_d^\sigma = (2, 0) - P_d. \tag{8.8}$$

Now let $f = 1$, so that $\Omega_1 = \Sigma$ is the Hilbert class field of K , and let $Q_K \in E(\Sigma)$ be the trace of the point P_d to the field K :

$$Q_K = \sum_{\sigma_1 \in G} P_d^{\sigma_1}, \quad G = \text{Gal}(\Sigma/K). \tag{8.9}$$

It is clear that Q_K has coordinates in K , since it is invariant under the group G . We shall prove that Q_K is a nontrivial point of $E(K)$, i.e. that Q_K does not lie in $E(\mathbb{Q}) = \{O, (0, 0), (2, 0), (-2, 0)\}$, whenever the class number $h(K)$ of the field K is odd.

Theorem 8.11. If $d = p \equiv 7 \pmod{8}$ is a prime, then the point Q_K is a nontrivial point on $E : Y^2 = X(X^2 - 4)$ which is defined over the field $K = \mathbb{Q}(\sqrt{-p})$.

Proof. First we apply the automorphism σ to the point Q_K :

$$\begin{aligned} Q_K^\sigma &= \sum_{\sigma_1 \in G} P_d^{\sigma_1 \sigma} = \sum_{\sigma_1 \in G} P_d^{\sigma \sigma_1^{-1}} \\ &= \sum_{\sigma_1 \in G} (2, 0) - \sum_{\sigma_1 \in G} P_d^{\sigma_1^{-1}} \end{aligned}$$

by (8.8) and (8.9), so that

$$Q_K^\sigma = [h(K)](2, 0) - Q_K. \quad (8.10)$$

Now assume that Q_K has coordinates in \mathbb{Q} . Then the last equation yields

$$2Q_K = [h(K)](2, 0) = (2, 0), \quad (8.11)$$

since the class number of K is odd. If $Q_K = (x, y)$, then the doubling formula for E gives that

$$\frac{1}{4} \frac{(x^2 + 4)^2}{x^3 - 4x} = 2,$$

and therefore $x = 2 \pm 2\sqrt{2} \notin \mathbb{Q}$. Hence, equation (8.11) is impossible and $Q_K \notin E(\mathbb{Q})$. \square

Remark. When $d = 23$, the point Q_K is computed to be

$$Q_K = \left(\frac{-7 + \sqrt{-23}}{9}, \frac{-50 + 2\sqrt{-23}}{27} \right).$$

Note that $[2]Q_K = (49/36, -77\sqrt{-23}/216)$, so that there is a solution over \mathbb{Q} of the twisted curve $-23Y^2 = X(X^2 - 4)$. A similar circumstance holds in the general case, since (8.10) implies that $([2]Q_K)^\sigma = -[2]Q_K$ and therefore $[2]Q_K = (x, y\sqrt{-p})$, with $x, y \in \mathbb{Q}$. An elementary argument similar to the above shows that $[2]Q_K \neq (0, 0), (-2, 0)$, so that $xy \neq 0$. This gives:

Corollary. If $p \equiv 7 \pmod{8}$ is a prime, the equation $E_p : pY^2 = X(X^2 - 4)$ has a nonzero rational solution.

Note that the above proof of Theorem 8.11 and its corollary is algebraic, except for the use of the properties of the Schläfli functions in Section 2. (See [9] for Deuring's algebraic derivation of the theorems of complex multiplication.)

For a general d with $d \equiv 7 \pmod{8}$ it is easy to see that the point $P_d = (\beta, 2(\beta - 2)\sqrt{\gamma})$ has infinite order on the curve E . This is because E has complex multiplication by $\mathbb{Z}[i]$, and is defined over \mathbb{Q} , so that the squares of the X -coordinates of torsion points on E generate abelian extensions of $k = \mathbb{Q}(i)$. If P_d had finite order, then $k(X(P_d)^2) = k(\beta^2) = \mathbb{Q}(i, \beta^2) = \Sigma(i)$, by Proposition 8.1, so $\Sigma(i) = \Sigma_4$ would be abelian over k . Since $\Sigma \cap k = \mathbb{Q}$, we have that $\text{Gal}(\Sigma_4/k) \cong \text{Gal}(\Sigma/\mathbb{Q})$, so Σ would be abelian over \mathbb{Q} . In that case, Σ coincides with its genus field, which corresponds in the sense of class field theory to the group of squares in the class group of K . Hence, the class group would have exponent 2. Then $\wp_2^2 \sim 1$ implies that $16 = x^2 + dy^2$ for $x, y \in \mathbb{Z}$, $x \equiv y \pmod{2}$, and $xy \neq 0$. It follows that $d = 7$ or 15 . In both of these cases it can be checked directly that P_d has infinite order on E . This proves part a) of the following theorem.

Theorem 8.12. a) For any $d \equiv 7 \pmod{8}$, the point $P_d = (\beta, 2(\beta - 2)\sqrt{\gamma})$ on E has infinite order.

b) If $d \equiv 7 \pmod{8}$, then the only torsion points on the curve E in $K = \mathbb{Q}(\sqrt{-d})$ are the points in $E(\mathbb{Q}) = E[2]$.

c) If the point $Q_K \notin \{O, (0, 0), (2, 0), (-2, 0)\}$, then Q_K has infinite order on E .

Proof of b). Assume that the point $Q \in E(K)$ has finite order, and that $Q \notin E(\mathbb{Q})$. Then the coordinates (x, y) of Q are algebraic integers, since the prime 2 splits in K (see [34, Thm. 7.1, p. 240]). Furthermore, we know $y^2 \mid -4^4$, so $y \mid 16 \cong \wp_2^4 \wp_2'^4$, by the arguments of [34, Cor. 7.2, p. 240]. Now y cannot be a rational integer, since the polynomial $X^3 - 4X - y^2$ is irreducible in the variable X over \mathbb{Q} for any integer y dividing 16, so its roots have degree 3 and cannot lie in K . Then the power of \wp_2 dividing y is not the same as the power of \wp_2' dividing y : without loss of generality $(y/2^r) = \wp_2^e \neq (1)$, with $1 \leq e+r \leq 4$. It follows that $2^{e+2} = u^2 + dv^2$, with $u \equiv v \equiv 1 \pmod{2}$. This implies that $d \leq 63$, so $d \in \{7, 15, 23, 31, 39, 47, 55\}$, from which 47 can be discarded, because the order of \wp_2 in the class group of $\mathbb{Q}(\sqrt{-47})$ is 5. This yields only the following solutions and their conjugates:

$$x = \frac{-1 + \sqrt{-7}}{2}, \quad y = \pm \frac{5 - \sqrt{-7}}{2} = \pm \left(\frac{-1 + \sqrt{-7}}{2} \right)^3;$$

Table 1: Points Q_K on $E(K)$ for $15 \leq d \leq 103$.

d	Q_K	d	Q_K
15	$(-3, \sqrt{-15})$	63	O
23	$\left(\frac{-7+\sqrt{-23}}{9}, \frac{-50+2\sqrt{-23}}{27}\right)$	71	$\left(\frac{12809-8183\sqrt{-71}}{29241}, \frac{13245170+2373070\sqrt{-71}}{5000211}\right)$
31	$\left(\frac{151+23\sqrt{-31}}{49}, \frac{1060+460\sqrt{-31}}{343}\right)$	79	$\left(\frac{-761-49\sqrt{-79}}{961}, \frac{53660+980\sqrt{-79}}{29791}\right)$
39	$\left(\frac{1}{13}, -\frac{15}{169}\sqrt{-39}\right)$	87	$(-27, 15\sqrt{-87})$
47	$\left(\frac{23+17\sqrt{-47}}{49}, \frac{-900+204\sqrt{-47}}{343}\right)$	95	$\left(\frac{9}{5}, \frac{3}{25}\sqrt{-95}\right)$
55	$\left(\frac{1}{5}, -\frac{3}{25}\sqrt{-55}\right)$	103	$\left(\frac{-1103+161\sqrt{-103}}{1681}, \frac{-151810+5474\sqrt{-103}}{68921}\right)$

and

$$x = 1 - \sqrt{-7}, \quad y = \pm 2^2 \left(\frac{1 + \sqrt{-7}}{2} \right).$$

In both cases $X(2Q) = 1/4$, so these points have infinite order on E . This proves parts b) and c). \square

Based on Theorem 8.12 and the examples in Table 1, we make the following conjecture.

Conjecture. If d is a positive, square-free integer with $d \equiv 7 \pmod{8}$ and $K = \mathbb{Q}(\sqrt{-d})$, then the point $Q_K \in E(K)$ defined by (8.9) has infinite order on the curve $E : Y^2 = X(X^2 - 4)$.

This conjecture holds for all d of the required type for $d < 100$. If true, the conjecture implies there are $\gtrsim \frac{1}{\pi^2}X$ positive, square-free integers $d \leq X$ for which the quadratic twist E_{-d} has rank ≥ 1 . (See Landau [23, p. 635] and [4].) We note that this curve is not one of the curves discussed by Byeon [4], since it has no nontrivial rational 3-torsion and conductor $N = 64$. Since the twist $E_{-d} \cong E_d$ is isomorphic to the curve $Y^2 = X(X^2 - 4d^2)$, a proof of the above conjecture would give an unconditional proof that square-free positive integers $2d \equiv 14 \pmod{16}$ are congruent numbers. (See Koblitz, *Introduction to Elliptic Curves and Modular Forms*, p. 92.)

9 Connection with supersingular curves.

We compare Theorem 7.1 with the result of Theorem 1.2(i) of [27], according to which the values of $\bar{\lambda}$ for which the Legendre normal form

$$E_{\bar{\lambda}} : Y^2 = X(X-1)(X-\bar{\lambda})$$

is supersingular in characteristic $p > 2$ are fourth powers in the finite field \mathbb{F}_{p^2} .

If \mathfrak{p} is a prime ideal of Ω_f lying above the odd, rational prime p for which the Legendre symbol $(-d/p) = 0$ or -1 , then the degree of \mathfrak{p} over \mathbb{Q} is either 1 or 2, so that $R_{\Omega_f}/\mathfrak{p} \cong \mathbb{F}_p$ or \mathbb{F}_{p^2} . Since -1 is a fourth power in \mathbb{F}_{p^2} , then for any λ satisfying the hypothesis of Theorem 7.1 the residue class $\bar{\lambda} = \lambda \pmod{\mathfrak{p}}$ will certainly be a fourth power in \mathbb{F}_{p^2} . By Deuring's reduction theory [7], any supersingular curve $E_{\bar{\lambda}}$ arises by reduction $(\text{mod } \mathfrak{p})$ from a curve E_{λ} with complex multiplication by R_{-d} , if $\sqrt{-d}$ injects into its endomorphism ring $\text{End}(E_{\bar{\lambda}})$. To show that Theorem 1.2(i) of [27] can be derived from Theorem 7.1, we prove the following theorem.

Theorem 9.1. If $E_{\bar{\lambda}}$ is a supersingular curve in characteristic $l \geq 3$, then there exists a positive integer $d \equiv 7 \pmod{8}$ for which $\sqrt{-d}$ injects into $\text{End}(E_{\bar{\lambda}})$.

Equivalently, let \mathbf{M} be any maximal order of the definite quaternion algebra D_l which is ramified only at l and the infinite prime p_{∞} . Then some square-root $\sqrt{-d}$ with positive $d \equiv 7 \pmod{8}$ injects into \mathbf{M} .

Remark. Note that if $\sqrt{-d} \rightarrow \mu$ injects into the maximal order \mathbf{M} , then $K = \mathbb{Q}(\sqrt{-d})$ is isomorphic to a maximal commutative subfield $\mathbb{Q}(\mu)$ of D_l , and is therefore a splitting field for D_l . By the well-known criterion for splitting fields of central simple algebras [re, Thm. 32.15, p. 278], it is clear that the prime l does not split in K , i.e., $\left(\frac{-d}{l}\right) = 0$ or -1 . Conversely, if this condition holds, then K is a splitting field for D_l and is isomorphic to a maximal subfield of D_l (see [29, Cor. 28.10, p. 240]).

Proof of Theorem 9.1. The assertion of Theorem 9.1 is equivalent to the statement that for some square-free integer $d_1 \equiv 7 \pmod{8}$ and some embedding $\mathbb{Q}(\sqrt{-d_1}) \rightarrow K' = \mathbb{Q}(\mu) \subset D_l$, the conductor t of the order $\mathcal{O} = \mathbf{M} \cap \mathbb{Q}(\mu)$ in the field K' is either odd or exactly divisible by 2. In the first case $t\sqrt{-d_1} = \sqrt{-t^2 d_1}$ injects into \mathbf{M} and $d = t^2 d_1 \equiv 7 \pmod{8}$; and in the second case the same situation holds with t replaced by $t/2$.

We shall use the following characterization of maximal orders in the quaternion algebra D_l first given by Dorman [11], and extended and clarified by Lauter and Viray [25, pp.13-19]. Let $l \geq 3$ be a prime and $d_1 = p \equiv 7 \pmod{8}$ a prime for which $\left(\frac{-p}{l}\right) = -1$. Consider the quaternion algebra $D_l = \left(\frac{-lq, -p}{\mathbb{Q}}\right)$, where q is a prime for which $-lq \equiv 1 \pmod{8}$ and $\left(\frac{-p}{q}\right) = +1$. The Hilbert symbol conditions

$$\left(\frac{-lq, -p}{r}\right) = +1, \text{ for all primes } r \neq l,$$

and

$$\left(\frac{-lq, -p}{l}\right) = \left(\frac{-lq, -p}{p_\infty}\right) = -1,$$

imply that D_l is ramified only at l and the infinite prime p_∞ . Furthermore, setting $L = \mathbb{Q}(\sqrt{-p})$, D_l is isomorphic to the algebra

$$\mathbb{B} = \{[\alpha, \beta] := \begin{pmatrix} \alpha & \beta \\ -lq\bar{\beta} & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in L\},$$

with the embedding $L \rightarrow \mathbb{B}$ defined by $\alpha \in L \rightarrow [\alpha, 0] \in \mathbb{B}$.

Now let $\mu = [\sqrt{-p}, 0] \in \mathbb{B}$ and let \mathbf{M} be any maximal order in \mathbb{B} . Assume that $\mathcal{O} = \mathbf{M} \cap \mathbb{Q}(\mu)$ has conductor t , where $\mu^2 = -p$. If t is odd or exactly divisible by 2, then by the above remarks there is nothing to show. So we assume $4 \mid t$, and we let $d = -t^2p$ be the discriminant of \mathcal{O} . Since q splits in L , we can write $q\mathcal{O} = \mathfrak{q}\bar{\mathfrak{q}}$, for a prime ideal \mathfrak{q} of \mathcal{O} . We may also assume $(q, t) = 1$, since we can replace q by any other odd prime satisfying the same conditions to obtain an isomorphic algebra.

By [25, Lemma 6.10] the maximal order \mathbf{M} is conjugate in \mathbb{B} (by an element in L^\times) to the maximal order $R(\mathfrak{a}, \lambda)$ determined as follows. The ideal \mathfrak{a} is some integral, invertible ideal of \mathcal{O} , relatively prime to the conductor t , and λ is an element of \mathcal{O} for which

- (1) $\lambda\mathfrak{q}^{-1}\bar{\mathfrak{a}}\mathfrak{a}^{-1} \subseteq \mathcal{O}$, and
- (2) $\mathbf{N}(\lambda) \equiv -lq \pmod{d}$.

With these definitions, and putting $\mathfrak{D} = (1/\sqrt{d})$, the set

$$R(\mathfrak{a}, \lambda) = \{[\alpha, \beta] \mid \alpha \in \mathfrak{D}^{-1}, \beta \in \mathfrak{q}^{-1}\mathfrak{D}^{-1}\bar{\mathfrak{a}}\mathfrak{a}^{-1}, \alpha - \lambda\beta \in \mathcal{O}\}$$

is a maximal order in \mathbb{B} with $R(\mathfrak{a}, \lambda) \cap \mathbb{Q}(\mu) = \mathcal{O}$. Now we take $\mu_1 = [0, \beta] \in R(\mathfrak{a}, \lambda)$, where $\beta \in \bar{\mathfrak{a}}$ is chosen as follows. Suppose the ideal $\bar{\mathfrak{a}}$ has the integral basis $\{a, \omega_1\}$

in \mathcal{O} , where $a = \mathbf{N}(\bar{a})$ is prime to t , $\omega_1 = x + \frac{ty}{2}\sqrt{-p}$, with $x, y \in \mathbb{Z}$, and $a \mid \mathbf{N}(\omega_1)$. Then we let

$$\beta = ra + 2s(x + \frac{ty}{2}\sqrt{-p}) \in \bar{a}, \quad r \equiv 1 \pmod{2}, \quad r, s \in \mathbb{Z}.$$

Then $\mathbf{N}(\beta) = (ra + 2sx)^2 + (sty)^2 p \equiv 1 \pmod{8}$, since $4 \mid t$. It follows that the minimal polynomial of $\mu_1 = [0, \beta]$ is $X^2 + lq\mathbf{N}(\beta)$, where $d_2 = lq\mathbf{N}(\beta) \equiv 7 \pmod{8}$, by the choices of q and β . Then $\mu_1^2 = -d_2$, so $\sqrt{-d_2}$ injects into $R(\mathfrak{a}, \lambda)$, and therefore also into the isomorphic maximal order \mathbf{M} . This proves the theorem. \square

This theorem verifies the remarks in the Introduction immediately following Theorem 1.3.

10 The solutions as values of modular functions.

The solution (π_f, ξ_f) of the Fermat equation that we have given in Theorems 1.1 and 6.5 can be represented in terms of modular functions. First recall that the Schläfli functions $\mathfrak{f}(\tau), \mathfrak{f}_1(\tau), \mathfrak{f}_2(\tau)$ are defined by

$$\mathfrak{f}(\tau) = e^{-\frac{\pi i}{24}} \frac{\eta\left(\frac{\tau+1}{2}\right)}{\eta(\tau)}, \quad \mathfrak{f}_1(\tau) = \frac{\eta\left(\frac{\tau}{2}\right)}{\eta(\tau)}, \quad \mathfrak{f}_2(\tau) = \sqrt{2} \frac{\eta(2\tau)}{\eta(\tau)},$$

where $\eta(\tau)$ is the Dedekind η -function ([5, p. 256], [32, p. 148]). These functions have the infinite product representations

$$\begin{aligned} \mathfrak{f}(\tau) &= q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}), & \mathfrak{f}_1(\tau) &= q^{-\frac{1}{48}} \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}}), \\ \mathfrak{f}_2(\tau) &= \sqrt{2} q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 + q^n), & q &= e^{2\pi i \tau}. \end{aligned}$$

From the defining formulas (2.7) and (1.3) we have

$$\begin{aligned} \alpha^4 &= -y(w)^2 = -\mathfrak{f}_1(w/2)^8 \mathfrak{f}_1(w)^8, \\ -\frac{\beta^4}{\alpha^4} &= \frac{-16}{\alpha^4 - 16} = \frac{16}{\mathfrak{f}_1(w/2)^8 \mathfrak{f}_1(w)^8 + 16}. \end{aligned} \tag{10.1}$$

Now we use the relations $\mathfrak{f}_1(w)\mathfrak{f}_2(w/2) = \sqrt{2}$ and $\mathfrak{f}_1(w/2)^8 + \mathfrak{f}_2(w/2)^8 = \mathfrak{f}(w/2)^8$ from [35, pp. 114-115] to write the last expression in (10.1) in the form

$$\frac{16}{\mathfrak{f}_1(w/2)^8 \mathfrak{f}_1(w)^8 + 16} = \frac{\mathfrak{f}_2(w/2)^8}{\mathfrak{f}_1(w/2)^8 + \mathfrak{f}_2(w/2)^8} = \frac{\mathfrak{f}_2(w/2)^8}{\mathfrak{f}(w/2)^8}.$$

Hence, we have

$$\pi = \frac{\beta}{\zeta_8^j \alpha} = i^a \frac{\mathfrak{f}_2(w/2)^2}{\mathfrak{f}(w/2)^2}, \quad w = \frac{v + \sqrt{-d}}{2}, \quad v^2 \equiv -d \pmod{16}. \quad (10.2)$$

It follows that

$$\frac{\beta^4}{16} = 1 + \frac{\beta^4}{\alpha^4} = 1 - \frac{\mathfrak{f}_2(w/2)^8}{\mathfrak{f}(w/2)^8} = \frac{\mathfrak{f}_1(w/2)^8}{\mathfrak{f}(w/2)^8}.$$

This gives

$$\xi = \frac{\beta}{2} = i^b \frac{\mathfrak{f}_1(w/2)^2}{\mathfrak{f}(w/2)^2}. \quad (10.3)$$

From numerical calculations it appears that the factor i^b in this formula is given by

$$i^b = i^{-v} = \begin{cases} i, & \text{if } v = 3 \text{ and } d \equiv 7 \pmod{16}, \\ -i, & \text{if } v = 1 \text{ and } d \equiv 15 \pmod{16}. \end{cases} \quad (10.4)$$

We will prove this at the end of this section.

From [36, p. 1646] or [35, p. 179] we know that $X = \mathfrak{f}^{24}(w/2), -\mathfrak{f}_1^{24}(w/2), -\mathfrak{f}_2^{24}(w/2)$ are the roots of

$$0 = (X - 16)^3 - j(w/2)X = (X - 16)^3 - \frac{(\beta^8 - 16\beta^4 + 256)^3}{\beta^8(\beta^4 - 16)^2}X,$$

where we have used (2.8) and the defining relation between α and β . The right side of the last equation factors, its roots being

$$X = -\frac{2^{12}}{\beta^4(\beta^4 - 16)}, \quad \frac{\beta^8}{\beta^4 - 16}, \quad -\frac{(\beta^4 - 16)^2}{\beta^4}.$$

By Lemma 6.6 and the arguments at the end of the proof of Theorem 8.6, the ideal factorizations of these three expressions in β are, respectively, 1 , $\wp_2'^{12}$, and \wp_2^{12} . It follows from (10.3) that we must have

$$\begin{aligned} \mathfrak{f}^{24}(w/2) &= -\frac{2^{12}}{\beta^4(\beta^4 - 16)}, \quad \mathfrak{f}_1^{24}(w/2) = -\frac{\beta^8}{\beta^4 - 16}, \\ \mathfrak{f}_2^{24}(w/2) &= \frac{(\beta^4 - 16)^2}{\beta^4}. \end{aligned}$$

Now, from the proof of Theorem 8.6 we have

$$-\beta(\beta^2 + 4) = \left(\frac{\beta^{\tau^2}}{2} \left(\frac{\beta}{2} - 1 \right) \right)^4, \quad (10.5a)$$

$$\beta^3(\beta^2 - 4) = \left(\frac{4\beta\beta^{\tau^2\sigma}}{\beta^{\tau^2}(\beta - 2)} \right)^4. \quad (10.5b)$$

These formulas give that

$$\mathfrak{f}_1^{24}(w/2) = \frac{\beta^{12}}{\beta^3(\beta^2 - 4) \times -\beta(\beta^2 + 4)} = \frac{\beta^{12}}{(\beta\beta^{\tau^2\sigma})^4},$$

so that

$$\mathfrak{f}_1^6(w/2) = \varepsilon_1 \frac{\beta^3}{\beta\beta^{\tau^2\sigma}} = \varepsilon_1 \frac{\alpha_1\beta}{2}, \quad (10.6a)$$

for some 4-th root of unity ε_1 . (This uses the fact that $\beta^\psi = 2\beta/\alpha_1$ for the automorphism $\psi = \tau^2\sigma$ in the proof of Theorem 8.5.) Similarly, we find that

$$\mathfrak{f}^6(w/2) = \varepsilon_2 \frac{2^3}{\beta\beta^{\tau^2\sigma}} = \varepsilon_2 \frac{4\alpha_1}{\beta^2} = \varepsilon_2 \frac{2}{\pi\xi}, \quad (10.6b)$$

where $\varepsilon_2^4 = 1$. These equations and (10.3) imply that $\varepsilon_1 = \varepsilon_2 i^b$. From (10.2) and (10.3) we also have that

$$\frac{\pi\xi}{2} = \frac{\beta^2}{4\zeta_8^j\alpha} = i^{a+b} \left(\frac{\mathfrak{f}_1(w/2)\mathfrak{f}_2(w/2)}{\sqrt{2}\mathfrak{f}(w/2)^2} \right)^2 = \frac{i^{a+b}}{\mathfrak{f}(w/2)^6},$$

where we have used the relation $\mathfrak{f}(\tau)\mathfrak{f}_1(\tau)\mathfrak{f}_2(\tau) = \sqrt{2}$ [35, p. 114]. Comparing with (10.6b) implies that

$$\varepsilon_2 = i^{a+b}, \quad \varepsilon_1 = i^{a+2b}.$$

It is clear from (10.6b) that $\mathfrak{f}(w/2)$ is a unit, and from (10.6a) that $(\mathfrak{f}_1^2(w/2)) = \wp_2'$.

We now use a result from the paper [36] of Yui and Zagier. We recall the definition from [36, p. 1647] of the Weber singular modulus for the primitive, positive definite quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ with discriminant $-d$: if τ_Q is the unique root of $Q(x, 1) = 0$ with positive imaginary part, then

$$f_w(Q) = \begin{cases} \zeta^{b(a-c-ac^2)} \mathfrak{f}(\tau_Q), & \text{if } (a, c) \equiv (0, 0) \pmod{2}; \\ \varepsilon_d \zeta^{b(a-c-ac^2)} \mathfrak{f}_1(\tau_Q), & \text{if } (a, c) \equiv (0, 1) \pmod{2}; \\ \varepsilon_d \zeta^{b(a-c+a^2c)} \mathfrak{f}_2(\tau_Q), & \text{if } (a, c) \equiv (1, 0) \pmod{2}; \end{cases}$$

where $\varepsilon_d = (-1)^{(-d-1)/8}$ and $\zeta = \zeta_{48} = e^{2\pi i/48}$. For Q we first take the form $Q_2(x, y) = 2x^2 - vxy + \left(\frac{v^2+d}{8}\right)y^2$, for which $Q_2(w/2, 1) = 0$, with $w = \frac{v + \sqrt{-d}}{2}$. The Weber singular modulus for Q_2 is

$$f_w(Q_2) = \zeta^{-v(2-(v^2+d)/8-(v^2+d)^2/32)} \mathfrak{f}(w/2). \quad (10.7)$$

If 3 does not divide d , then by the proposition of [36, p. 1647] we have $f_w(Q_2) \in \mathbb{Q}(j(w/2)) \subset \Omega_f$. We require a modest extension of this result.

Proposition 10.1. a) If $(d, 3) = 1$, then for any primitive quadratic form $Q(x, y) = ax^2 + bxy + cy^2$ of discriminant $-d \equiv 1 \pmod{8}$, the value $f_w(Q)$ depends only on the $SL_2(\mathbb{Z})$ equivalence class $\mathcal{A} = [Q]$ of Q and lies in $\mathbb{Q}(j(\tau_Q))$.

b) If $3 \mid d$, then the value $f_w(Q)^3$ depends only on the $SL_2(\mathbb{Z})$ equivalence class $\mathcal{A} = [Q]$ of Q and lies in $\mathbb{Q}(j(\tau_Q))$.

Proof. Part a) is proved in [36]. For part b), it is only necessary to modify the proof in [36] slightly. The proof that $f_w(Q)^3$ is an invariant of the class $\mathcal{A} = [Q]$ is the same as the proof in [36], except that only congruences (mod 16) need to be considered instead of congruences (mod 48). For the second part of the proof, choose $Q = [a, b, c]$ in $\mathcal{A} = [Q]$ so that $(a, 6) = 1$ is odd and $b \equiv -a \pmod{48}$. Then with the same notation as in [36], $a^2 \equiv 1 \pmod{24}$ implies that

$$f_w(\mathcal{A})^3 = \pm(\zeta^{b(a-c+a^2c)} \mathfrak{f}_2(\tau_Q))^3 = \pm(\zeta^{-1} \mathfrak{f}_2(\tau_Q))^3 = \pm \left(\frac{\sqrt{2}}{\mathfrak{f}(2\tau_Q - 1)} \right)^3.$$

Now $\varpi = 2\tau_Q - 1$ is a root of $\tilde{Q}(x, 1) = 0$ for the quadratic form

$$\tilde{Q} = Ax^2 + 2Bxy + Cy^2, \quad A = a, \quad B = a + b, \quad C = a + 2b + 4c.$$

(This corrects the value given for C in the last paragraph of the proof in [36, p. 1648].) By Thm. 6.4.1 of [32, p. 148], which is applicable since A is odd and $B \equiv 0 \pmod{16}$, it follows that $(\mathfrak{f}(\varpi)/\sqrt{2})^3 \in \mathbb{Q}(j(\varpi))$. But the discriminant of \tilde{Q} is $-4d$, so $j(\varpi)$ generates a subfield of the ring class field Ω_{2f} of K , which coincides with Ω_f . Hence, $f_w(\mathcal{A})^3 = f_w(Q)^3 \in \Omega_f$ has degree at most $h(-d)$. Since $\mathbb{Q}(j(\tau_Q)) \subseteq \mathbb{Q}(f_w(Q)^3)$ and $j(\tau_Q)$ has degree $h(-d)$, it follows that $\mathbb{Q}(f_w(Q)^3) = \mathbb{Q}(j(\tau_Q))$, which is what we needed to show. \square

We apply this proposition by raising (10.7) to the 6-th power:

$$f_w(Q_2)^6 = i^{-v(1-(v^2+d)/16-(v^2+d)^2/64)} \mathfrak{f}^6(w/2) = i^{-v(1-(v^2+d)/16)} \cdot i^{a+b} \frac{4\alpha_1}{\beta^2},$$

where the second equality follows from (10.6b) and $\frac{(v^2+d)^2}{64} \equiv 0 \pmod{4}$. Since $i \notin \Omega_f$ and both $4\alpha_1/\beta^2$ and $f_w(Q_2)^6$ are squares in Ω_f , this implies

$$\frac{4\alpha_1}{\beta^2} = f_w(Q_2)^6, \tag{10.8}$$

and

$$a + b \equiv v \left(1 - \frac{v^2 + d}{16} \right) \pmod{4}. \tag{10.9}$$

This proves the following theorem.

Theorem 10.2. If $(d, 3) = 1$ and $f_w(Q_2)$ is defined by (10.7), then $\frac{4\alpha_1}{\beta^2}$ is the 6-th power of the unit $f_w(Q_2)$ in Ω_f . If $3 \mid d$, then $\frac{4\alpha_1}{\beta^2}$ is the square of the unit $f_w(Q_2)^3$ in Ω_f .

Note that $4\alpha_1/\beta^2 = 2^3/(\beta\beta^{\tau^2\sigma})$ lies in the fixed field of the automorphism $\psi = \tau^2\sigma$, and so has degree at most $h(-d)$ over \mathbb{Q} . Since $j(w/2)$ is a rational function of $\mathfrak{f}^{24}(w/2)$, it follows from (10.6b) that $\mathbb{Q}(j(w/2)) \subseteq \mathbb{Q}(4\alpha_1/\beta^2)$ and therefore $\mathbb{Q}(j(w/2)) = \mathbb{Q}(4\alpha_1/\beta^2)$, since the degree of $j(w/2)$ equals $h(-d)$. Thus,

$$\mathbb{Q}(j(w/2)) = \mathbb{Q}(4\alpha_1/\beta^2) = \mathbb{Q}(f_w(Q_2)^e) = \text{Fix}(\psi),$$

where $e = 1$ or 3 and $Fix(\psi)$ denotes the fixed field of $\psi = \tau^2\sigma = \tau\sigma\tau^{-1}$ inside Ω_f . This agrees with the fact that $\mathbb{Q}(j(w)) = Fix(\sigma)$ (see the proof of Proposition 8.3) and that $\mathbb{Q}(j(w/2)) = \mathbb{Q}(j(w))^{\tau^{-1}}$.

Remark. Equation (10.8) clarifies the relationship between the generators of Ω_f considered by Yui and Zagier in [36] and the algebraic numbers $\alpha_1 = \zeta_8^j\alpha$ and β that are the main focus of this paper. It also makes it clear that $4\alpha_1/\beta^2$ is a natural unit to consider in the case that 3 does divide d . See Case **B** in [36, p. 1660] and the example $d = 159$ in Section 12.

Next, we consider the quadratic form $Q_1(x, y) = x^2 - vxy + \left(\frac{v^2+d}{4}\right)y^2$, for which $Q_1(w, 1) = 0$. The formulas of Yui and Zagier give that

$$f_w(Q_1) = (-1)^{(-d-1)/8} \zeta^{-v} \mathfrak{f}_2(w). \quad (10.10)$$

The value $f_w(Q_1)$ (or $f_w(Q_1)^3$) is real since $j(w)$ is real, as the j -invariant of the principal class in R_{-d} . To find the connection with our generators, we use (8.0) to solve the equation

$$0 = (X - 16)^3 - j(w)X = (X - 16)^3 - \frac{(\beta^8 + 224\beta^4 + 256)^3}{\beta^4(\beta^4 - 16)^4}X.$$

Once again, this cubic factors, giving the roots

$$X = -\frac{(\beta^2 - 4)^4}{\beta^2(\beta^2 + 4)^2}, \quad \frac{(\beta^2 + 4)^4}{\beta^2(\beta^2 - 4)^2}, \quad -\frac{2^{12}\beta^4}{(\beta^4 - 16)^2}.$$

Using Lemma 6.6 and the computations at the end of the proof of Theorem 8.6, we find that these three roots generate the respective ideals in R_{Ω_f} :

$$\left(-\frac{(\beta^2 - 4)^4}{\beta^2(\beta^2 + 4)^2}\right) = \wp_2^{12}, \quad \left(\frac{(\beta^2 + 4)^4}{\beta^2(\beta^2 - 4)^2}\right) = (1), \quad \left(-\frac{2^{12}\beta^4}{(\beta^4 - 16)^2}\right) = (\wp_2')^{12}.$$

Yui and Zagier prove in [36] that $f_w(Q_1)^2$ and $f_w(Q_2)^2$ are conjugates over \mathbb{Q} (the same argument applies to $f_w(Q_1)^6$ and $f_w(Q_2)^6$ when $3 \mid d$), so $\mathfrak{f}_2(w)$ must be a unit, which forces

$$\mathfrak{f}_2^{24}(w) = -\frac{(\beta^2 + 4)^4}{\beta^2(\beta^2 - 4)^2} = -\frac{\beta^4(\beta^2 + 4)^4}{\beta^6(\beta^2 - 4)^2}.$$

Using (10.5) yields that

$$\mathfrak{f}_2^3(w) = \frac{\zeta_{16}^r}{2^6} \frac{\left(\beta^{\tau^2}(\beta - 2)\right)^3}{\beta\beta^{\tau^2\sigma}}, \quad (10.11)$$

for some *odd* integer r , since the 8-th power of ζ_{16}^r must give -1 . On the other hand, we have from (10.6b) and (10.8) that $\beta\beta^{\tau^2\sigma} = 2^3/f_w(Q_2)^6$. Putting this into (10.11) yields

$$\mathfrak{f}_2(w) = \frac{\zeta_{48}^{r'}}{2^3} \beta^{\tau^2} (\beta - 2) f_w(Q_2)^2,$$

where r' is an odd integer, and therefore

$$f_w(Q_1) = (-1)^{(-d-1)/8} \zeta_{48}^{r'-v} \times \frac{1}{2^3} \beta^{\tau^2} (\beta - 2) f_w(Q_2)^2.$$

If $d \not\equiv 0 \pmod{3}$, this implies that $r' \equiv v \pmod{24}$, since the cube roots of unity are not contained in Ω_f , and therefore

$$f_w(Q_1) = (-1)^{(-d-1)/8+(r'-v)/24} \frac{1}{2^3} \beta^{\tau^2} (\beta - 2) f_w(Q_2)^2. \quad (10.12a)$$

If $3 \mid d$, then on cubing we obtain instead that $r' \equiv v \pmod{8}$ and

$$f_w(Q_1)^3 = (-1)^{(-d-1)/8+(r'-v)/8} \left(\frac{1}{2^3} \beta^{\tau^2} (\beta - 2) \right)^3 f_w(Q_2)^6. \quad (10.12b)$$

These equations express relations between the units $f_w(Q_1)$, $f_w(Q_2)$, and $\xi^{\tau^2}(\xi - 1)/2$. Note that the last unit is one-half the Y -coordinate of the point $(-\beta, \xi^{\tau^2}(\xi - 1))$ on the curve $Y^4 = X(X^2 + 4)$. (See (8.5).)

There are similar formulas for the quadratic forms

$$Q_c(x, y) = cx^2 - vxy + \left(\frac{v^2 + d}{4c} \right) y^2,$$

$$Q_{2c}(x, y) = 2cx^2 - vxy + \left(\frac{v^2 + d}{8c} \right) y^2,$$

where $(c, 6d) = 1$, v is odd, and $16c \mid (v^2 + d)$. These are the forms corresponding to the ideals $\mathfrak{c} = (c, cw_1)$ and $\mathfrak{c}\wp_{2,-d} = (2c, cw_1)$, where $w_1 = w/c = (v + \sqrt{-d})/2c$. For these forms the Yui-Zagier formulas give

$$\begin{aligned} f_w(Q_c) &= (-1)^{(-d-1)/8} \zeta^{-v(c+(c^2-1)(v^2+d)/(4c))} \mathfrak{f}_2(w/c) \\ &= (-1)^{(-d-1)/8} \zeta^{-vc} \mathfrak{f}_2(w/c), \quad (c, 6d) = 1; \end{aligned} \quad (10.13)$$

$$\begin{aligned}
f_w(Q_{2c}) &= \zeta^{-v(2c-(v^2+d)/(8c)-(v^2+d)^2/(32c))} \mathfrak{f}(w/(2c)) \\
&= \zeta^{-2vc(c^2-(v^2+d)/16-(v^2+d)^2/64)} \mathfrak{f}(w/(2c)) \\
&= \zeta^{-2vc(1-(v^2+d)/16-(v^2+d)^2/64)} \mathfrak{f}(w/(2c)), \tag{10.14}
\end{aligned}$$

where the next to last equality follows from multiplying inside and outside the parenthesis in the exponent by c , which is valid since $c^2 \equiv 1 \pmod{24}$.

Further, let $\tau_c = \left(\frac{\Omega_f/K}{R_K \mathfrak{c}}\right)$ (note that $\tilde{\mathfrak{c}} = R_K \mathfrak{c}$ is the ideal of R_K corresponding to the ideal \mathfrak{c} of R_{-d}), so that

$$j(w)^{\tau_c^{-1}} = j(\mathfrak{c}) = j(w_1)$$

and

$$j(w/2)^{\tau_c^{-1}} = j(\wp_{2,-d} \mathfrak{c}) = j(w_1/2).$$

Then, defining

$$\alpha_c = \alpha_1^{\tau_c^{-1}}, \quad \beta_c = \beta^{\tau_c^{-1}},$$

we have all the same formulas for α_c, β_c from Sections 6-8 that we have for α_1, β . In place of the automorphism σ in those formulas we need to take its conjugate $\sigma_c = \tau_c \sigma \tau_c^{-1}$, since

$$\frac{2(\beta_c + 2)}{\beta_c - 2} = \beta^{\sigma \tau_c^{-1}} = \beta_c^{\tau_c \sigma \tau_c^{-1}}.$$

Furthermore, the formulas of this section all hold when w is replaced by $w_1 = w/c$.

Now we apply the Reciprocity Law of Shimura [32, p. 123 and p. 72] to the algebraic numbers $\mathfrak{f}(w/2)$ and $\mathfrak{f}_2(w)$ in (10.7) and (10.10). For this we note that the coefficients in the q -expansions of $\mathfrak{f}(\tau)$ and $\mathfrak{f}_2(\tau)$ are in \mathbb{Q} and $\mathbb{Q}(\sqrt{2})$, respectively, and these are modular functions for $\Gamma(48)$ (see [36] and [32, p. 148]). Taking the matrix C in the Reciprocity Law to be

$$C = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \quad \text{with } cC^{-1} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} = -TCT,$$

and $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we have that $C(w, 1)^t$ is a basis of the ideal \mathfrak{c} , and

$$\mathfrak{f} \circ cC^{-1} = \mathfrak{f}, \quad \mathfrak{f}_2 \circ cC^{-1} = \mathfrak{f}_2,$$

in Schertz's notation [32, p. 72]. (The second formula holds because $\mathfrak{f}_2(-1/\tau) = \mathfrak{f}_1(\tau)$ and the latter has a q -series with rational coefficients.) Now the Frobenius automorphism for the conjugate ideal $\bar{\mathfrak{c}}$ satisfies $(\Omega_f/K, R_K \bar{\mathfrak{c}}) = \tau_c^{-1}$. If L is an abelian

extension of K containing the numbers $\zeta = \zeta_{48}$, $\mathfrak{f}(w/2)$, and $\mathfrak{f}_2(w)$ ($L = \Sigma_{48f}(\zeta_{48})$ would suffice by [32, Theorem 5.2.1]), the Frobenius automorphism $(L/K, R_K \bar{\mathfrak{c}})$ extends the automorphism τ_c^{-1} on L/K . Denoting $(L/K, R_K \bar{\mathfrak{c}})$ by τ_c^{-1} , the Reciprocity Law implies

$$\mathfrak{f}(w/2)^{\tau_c^{-1}} = \mathfrak{f}(w/(2c)) = \mathfrak{f}(w_1/2), \quad \mathfrak{f}_2(w)^{\tau_c^{-1}} = \mathfrak{f}_2(w/c) = \mathfrak{f}_2(w_1).$$

Furthermore, $\zeta^{\tau_c^{-1}} = \zeta^{Norm(\bar{\mathfrak{c}})} = \zeta^c$ and equations (10.7), (10.10), (10.13), and (10.14) give

$$f_w(Q_2)^{\tau_c^{-1}} = f_w(Q_{2c}), \quad f_w(Q_1)^{\tau_c^{-1}} = f_w(Q_c). \quad (10.15)$$

Now it is easy to see that every ideal class in R_{-d} contains an ideal \mathfrak{c} with $(Norm(\mathfrak{c}), 6d) = 1$, given by $\mathfrak{c} = (c, (v + \sqrt{-d})/2)$, with $16c \mid (v^2 + d)$. Furthermore, $(1, (v + \sqrt{-d})/2) = (1)$, so that $f_w(Q_1)$ is the singular modulus for the principal class, as above. From (10.15) it follows that $f_w(Q_1)$ is conjugate to the Weber singular modulus of every other ideal class.

This verifies the conjecture of [36, p. 1648] that the numbers $f_w(\mathcal{A}) = f_w(Q)$ are algebraic conjugates of each other, as \mathcal{A} varies over ideal classes (or equivalence classes of quadratic forms), in the case that 3 does not divide d . By Proposition 10.1b) and the above computations we get a similar statement when $3 \mid d$. (Note that $f_w(Q_2)^e$, for $e = 1$ or 3 , has $h(-d)$ conjugates since $\mathbb{Q}(f_w(Q_2)^e) = \mathbb{Q}(j(w/2))$.)

Theorem 10.3. a) If $(d, 3) = 1$ and $\mathcal{O} = R_{-d}$, then the polynomial

$$W_{-d}(X) = \prod_{\mathcal{A} \in Pic(\mathcal{O})} (X - f_w(\mathcal{A}))$$

is the minimal polynomial of $f_w(Q_1)$ over \mathbb{Q} .

b) If $3 \mid d$, then

$$\tilde{W}_{-d}(X) = \prod_{\mathcal{A} \in Pic(\mathcal{O})} (X - f_w(\mathcal{A})^3)$$

is the minimal polynomial of $f_w(Q_1)^3$ over \mathbb{Q} .

If we now apply the automorphism τ_c^{-1} to (10.12a) and (10.12b), we obtain:

Theorem 10.4. The Weber singular moduli for the quadratic forms $Q_c = [c, -v, \frac{v^2+d}{4c}]$ and $Q_{2c} = [2c, -v, \frac{v^2+d}{8c}]$ corresponding to the ideals \mathfrak{c} and $\wp_{2,-d}\mathfrak{c}$, with $c = Norm(\mathfrak{c})$ and $(c, 6d) = 1$, are related by the formulas

$$f_w(Q_c) = \frac{\epsilon}{2^3} \beta_c^{\tau^2} (\beta_c - 2) f_w(Q_{2c})^2, \quad \text{if } (d, 3) = 1,$$

$$f_w(Q_c)^3 = \frac{\epsilon}{2^9} (\beta_c^{\tau^2} (\beta_c - 2))^3 f_w(Q_{2c})^6, \quad \text{if } 3 \mid d,$$

where $\epsilon = (-1)^{(-d-1)/8+e(r'-v)/24}$ ($e = 1$ or 3) is independent of c and $\beta_c = \beta^{\tau_c^{-1}}$, with $\tau_c = (\Omega_f/K, R_K \mathfrak{c})$.

As a corollary of this theorem, we have:

Theorem 10.5. Assume that $(d, 3) = 1$. If \mathcal{A} represents the ideal class containing \mathfrak{c} in the order R_{-d} , and \mathcal{T} is the class containing the ideal $\wp_{2,-d} = \wp_2 \cap R_{-d}$, then

$$2 \frac{f_w(\mathcal{A})}{f_w(\mathcal{T}\mathcal{A})^2} = \frac{\epsilon}{4} \beta_c^{\tau^2} (\beta_c - 2)$$

is the Y -coordinate of the point $\Phi = (-\beta_c, \frac{\epsilon}{4} \beta_c^{\tau^2} (\beta_c - 2))$ on the curve $Y^4 = X(X^2 + 4)$.

The formulas in (10.15) show that the Weber singular moduli $f_w(Q)$ transform under the automorphisms of $Gal(\Omega_f/K)$ in the same way that the corresponding j -invariants transform. Therefore, $j(w)^{\tau^{-1}} = j(w/2)$ and $j(w)^{\tau^{-2}} = j(w/4)$ imply that

$$\begin{aligned} f_w(Q_1)^{\tau^{-1}} &= f_w(Q_2), \quad f_w(Q_1)^{\tau^{-1}\tau_c^{-1}} = f_w(Q_{2c}), \\ f_w(Q_1)^{\tau^{-2}} &= f_w(Q_4), \quad Q_4(x, y) = 4x^2 - vxy + \frac{v^2 + d}{16}. \end{aligned}$$

We are now ready to verify the formula (10.4).

Theorem 10.6. Equations (10.3)-(10.4) hold. Furthermore, $\zeta_8^{(j-1)/2} \frac{\eta(w/4)}{\eta(w)} \in \Omega_f$, where

$$j \equiv \begin{cases} \frac{3d+19}{8} = 3 \left(\frac{d+1}{8} \right) + 2 \pmod{8}, & \text{if } v = 3 \text{ and } d \equiv 7 \pmod{16}, \\ \frac{d+25}{8} = \frac{d+1}{8} + 3 \pmod{8}, & \text{if } v = 1 \text{ and } d \equiv 15 \pmod{16}. \end{cases} \quad (10.16)$$

Proof. We first prove that (10.4) holds. Note that β is determined among the fourth roots of β^4 by the condition that $\beta^\sigma = \frac{2(\beta+2)}{\beta-2} = \bar{\beta}$ is the complex conjugate of β . This is because $j(w)$ is fixed by σ : since $j(w) = j(\mathcal{O})$ is the j -invariant corresponding

to the principal class $\mathcal{O} = \mathbf{R}_{-d}$, it is real, and because $Fix(\sigma) = \mathbb{Q}(j(w))$ it follows that σ is complex conjugation on Ω_f . This implies that the equation

$$\beta\bar{\beta} = 2(\beta + \bar{\beta}) + 4$$

holds. Now this condition holds for β , and we claim that it cannot also hold for $-\beta$, $i\beta$, or $-i\beta$. If it also held for $-\beta$ in place of β then we would have that $\beta + \bar{\beta} = 0$, which is impossible, since $\bar{\beta}$ is a \mathbb{Q} -conjugate of β , but $-\beta$ is not (see Proposition 8.1). If it held for $i\beta$ in place of β , then we would have $2(\beta + \bar{\beta}) = 2i(\beta - \bar{\beta})$, so $i = (\beta + \bar{\beta})/(\beta - \bar{\beta}) \in \Omega_f$, which is impossible. The same argument applies to $-i\beta$. Therefore, the factor i^b in (10.3) is determined by the condition that

$$\frac{i^b \mathfrak{f}_1(w/2)^2 + \mathfrak{f}(w/2)^2}{i^b \mathfrak{f}_1(w/2)^2 - \mathfrak{f}(w/2)^2} = \frac{i^{-b} \mathfrak{f}_1(-\bar{w}/2)^2}{\mathfrak{f}(-\bar{w}/2)^2}, \quad (10.17)$$

where we have used the fact that \mathfrak{f} and \mathfrak{f}_1 have q -expansions with rational coefficients.

Now we let $q_1 = q^{1/2} = e^{\pi i w/2} = e^{\pi i(v+i\sqrt{d})/4}$, so that $\bar{q}_1 = e^{-\pi i v/2} q_1 = i^{-v} q_1$, and write

$$q^{\frac{1}{48}} \mathfrak{f}_1(w/2) = A(q_1) = \prod_{n=1}^{\infty} (1 - q_1^{2n-1})$$

and

$$q^{\frac{1}{48}} \mathfrak{f}(w/2) = A(-q_1) = \prod_{n=1}^{\infty} (1 + q_1^{2n-1}).$$

With this notation, (10.17) is equivalent to the relation

$$(i^b A(q_1)^2 + A(-q_1)^2) A(-i^{-v} q_1)^2 = (A(q_1)^2 - i^{-b} A(-q_1)^2) A(i^{-v} q_1)^2. \quad (10.18)$$

Now, for $v = 3$, say, and an infinite sequence of integers $d \equiv 7 \pmod{16}$, we know that (10.18) holds for a fixed choice of the factor i^b . For this sequence of d 's we have $q_1 \rightarrow 0$ as $d \rightarrow \infty$. Since $A(q_1)$ is holomorphic in q_1 for $|q_1| < 1$, we thus have an identity in q_1 :

$$(i^b A(q_1)^2 + A(-q_1)^2) A(-iq_1)^2 = (A(q_1)^2 - i^{-b} A(-q_1)^2) A(iq_1)^2.$$

Now consider the first two terms on both sides of this equation, with $\lambda = i^b$:

$$1 + \lambda + (-2\lambda + 2 + 2i(\lambda + 1))q_1 + O(q_1^2) = 1 - \frac{1}{\lambda} + (-2 - \frac{2}{\lambda} - 2i(1 - \frac{1}{\lambda}))q_1 + O(q_1^2).$$

Equating coefficients of q_1 gives that $\lambda = i$, and therefore $i^b = i^{-v}$ when $v = 3$. This proves the identity

$$(iA(q_1)^2 + A(-q_1)^2)A(-iq_1)^2 = (A(q_1)^2 + iA(-q_1)^2)A(iq_1)^2,$$

which is equivalent to the assertion that the left side of this equation is an even function of q_1 . Taking the complex conjugate of this equation now shows that when $v = 1$, (10.18) holds with $i^b = -i = i^{-v}$. This proves (10.4).

Taking $v = 1$ or 3 and $b = -v$ in (10.9) implies

$$a \equiv v \left(2 - \frac{v^2 + d}{16} \right) \pmod{4},$$

or

$$a \equiv \begin{cases} \frac{-3d+5}{16} \pmod{4}, & \text{if } v = 3 \text{ and } d \equiv 7 \pmod{16}, \\ \frac{-d+31}{16} \pmod{4}, & \text{if } v = 1 \text{ and } d \equiv 15 \pmod{16}. \end{cases} \quad (10.19)$$

Now

$$\zeta_8^j \alpha = 2 \frac{\beta/2}{\beta/\zeta_8^j \alpha} = 2i^b \frac{\mathfrak{f}_1(w/2)^2}{\mathfrak{f}(w/2)^2} \times i^{-a} \frac{\mathfrak{f}(w/2)^2}{\mathfrak{f}_2(w/2)^2} = 2i^{b-a} \frac{\mathfrak{f}_1(w/2)^2}{\mathfrak{f}_2(w/2)^2}. \quad (10.20)$$

Expressing the last quotient in this equation in terms of the η -function gives

$$\zeta_8^j \alpha = 2i^{b-a} \frac{\eta(w/4)^2}{\eta(w/2)^2} \times \frac{\eta(w/2)^2}{2\eta(w)^2} = i^{b-a} \frac{\eta(w/4)^2}{\eta(w)^2}.$$

From the definition of α in (2.7) we conclude that $\zeta_8^{j-1} = i^{b-a}$, whence we have that $j \equiv 2(b-a) + 1 \pmod{8}$. Then (10.4) and (10.19) yield (10.16). \square

The above proof shows that identity (10.18) holds with $i^b = i^{-v}$ in all cases, with $w = (v + \sqrt{-d})/2$, whether or not v is restricted to be 1 or 3. Thus we have:

Theorem 10.7. If $w = (v + \sqrt{-d})/2$, with $16c \mid (v^2 + d)$ and $(c, 6d) = 1$, then

$$\frac{\beta_c}{2} = \frac{\beta_c^{-1}}{2} = i^{-vc} \frac{\mathfrak{f}_1(w/(2c))^2}{\mathfrak{f}(w/(2c))^2}.$$

The numbers β_c are class invariants. Namely, if $\mathfrak{c}_1 = (c_1, (v_1 + \sqrt{-d})/2)$ and $\mathfrak{c}_2 = (c_2, (v_2 + \sqrt{-d})/2)$ are ideals of \mathbf{R}_{-d} with norms c_1 and c_2 satisfying

$$16c_i \mid (v_i^2 + d), \quad (c_1 c_2, 6d) = 1, \quad v_1 \equiv v_2 \pmod{8},$$

then $\beta_{c_1} = \beta_{c_2}$ if and only if $\mathbf{c}_1 \sim \mathbf{c}_2$ in the ring R_{-d} .

Proof. The formula for β_c follows from the above remarks. For the proof that the numbers β_c are class invariants, let \mathbf{c}_1 and \mathbf{c}_2 be as in the statement of the theorem. Note first that the condition $v_1 \equiv v_2 \pmod{8}$ guarantees that the ideal $\wp_2 = (2, (v_i + \sqrt{-d})/2)$ (for $i = 1, 2$) is consistently determined. Actually, it would be enough to assume that $v_1 \equiv v_2 \pmod{4}$, but then

$$v_1^2 \equiv (v_2 + 4k)^2 \equiv v_2^2 + 8kv_2 \equiv v_1^2 + 8kv_2 \pmod{16}$$

implies the stronger condition that $8 \mid (v_1 - v_2)$. With this condition we have that

$$\frac{\beta}{2} = i^{-v_1} \frac{\mathfrak{f}_1((v_1 + \sqrt{-d})/2)^2}{\mathfrak{f}((v_1 + \sqrt{-d})/2)^2} = i^{-v_1} \frac{\mathfrak{f}_1((v_2 + \sqrt{-d})/2 + 4k)^2}{\mathfrak{f}((v_2 + \sqrt{-d})/2 + 4k)^2}.$$

By the transformation formulas for the Schläfli functions (see [36, p. 1647]), this becomes

$$\frac{\beta}{2} = i^{-v_2} \frac{\zeta^{-8k} \mathfrak{f}_1((v_2 + \sqrt{-d})/2)^2}{\zeta^{-8k} \mathfrak{f}((v_2 + \sqrt{-d})/2)^2} = i^{-v_2} \frac{\mathfrak{f}_1((v_2 + \sqrt{-d})/2)^2}{\mathfrak{f}((v_2 + \sqrt{-d})/2)^2},$$

so that the value of β does not depend on the choice of v_i . If $\beta_{c_1} = \beta_{c_2}$, then by (8.0) and $j(w)^{\tau_{c_i}^{-1}} = j(\mathbf{c}_i)$ we have $j(\mathbf{c}_1) = j(\mathbf{c}_2)$, which implies $\mathbf{c}_1 \sim \mathbf{c}_2$ in R_{-d} . Conversely, assume that $\mathbf{c}_1 \sim \mathbf{c}_2$. Then $\tau_{c_1} = \tau_{c_2}$ and $\beta_{c_1} = \beta^{\tau_{c_1}^{-1}} = \beta^{\tau_{c_2}^{-1}} = \beta_{c_2}$. This proves the theorem. \square

The proof of Theorem 10.6 also allows us to give an explicit formula for the unit γ and its square-root.

Proposition 10.8. If $w = (v + \sqrt{-d})/2$, with $16 \mid (v^2 + d)$, then

$$\sqrt{\gamma} = \pm \frac{1}{\sqrt{2}} \frac{\mathfrak{f}_1(w/2) \mathfrak{f}_1(-\bar{w}/2)}{\mathfrak{f}(w/2) \mathfrak{f}(-\bar{w}/2)}.$$

Thus, this expression lies in the field $\mathbb{Q}(j(w))$.

Proof. This follows from

$$\gamma = \frac{1}{2} \frac{\beta}{2} \frac{\beta + 2}{\beta - 2} = \frac{1}{2} i^{-v} \frac{\mathfrak{f}_1(w/2)^2}{\mathfrak{f}(w/2)^2} i^v \frac{\mathfrak{f}_1(-\bar{w}/2)^2}{\mathfrak{f}(-\bar{w}/2)^2},$$

by (10.3) and (10.17).

11 Generators of Ω_f/K when $3 \mid d$.

When $(d, 3) = 1$, the Weber singular moduli $f_w(Q)$ from [36] are generators of small height of the ring class fields Ω_f . However, when $3 \mid d$, the generators given by Yui and Zagier are not quite as good, in terms of the size of their discriminants, as is illustrated by the examples $d = 87, 159, 231$ given in [36, p. 1660]. These generators are essentially the numbers $f_w(Q)^3$ (when $3 \mid h(-d)$) and $f_w(Q)/\varepsilon^{\pm 1/3}$ (when $h(-d) \equiv \pm 1 \pmod{3}$), where ε is a fundamental unit of the field $\mathbb{Q}(\sqrt{d/3})$. However, the latter generators remained conjectural in [36].

In the case that $3 \mid d$, Theorem 10.4 gives us a systematic way of finding generators for Ω_f of small height, independent of whether $3 \mid h(-d)$ or not. We first multiply both sides of (10.12b) by $f_w(Q_2)^3$. This gives

$$f_w(Q_1)^3 f_w(Q_2)^3 = \frac{\epsilon}{29} (\beta^{\tau^2} (\beta - 2))^3 f_w(Q_2)^9.$$

The right side of this equation is a cube in Ω_f , so taking cube roots yields

$$f_w(Q_1) f_w(Q_2) = \frac{\epsilon \omega^l}{23} (\beta^{\tau^2} (\beta - 2)) f_w(Q_2)^3, \quad (11.1)$$

for some l , where $\omega = e^{2\pi i/3}$. Thus, $f_w(Q_1) f_w(Q_2) \in \Omega_f$, and by (10.15) we have

$$f_w(Q_c) f_w(Q_{2c}) = \frac{\epsilon \omega^{lc}}{23} (\beta_c^{\tau^2} (\beta_c - 2)) f_w(Q_{2c})^3.$$

It is easy to see that $f_w(Q_1) f_w(Q_2) = f_w(Q_1)^{1+\tau^{-1}}$ has degree at most $h(-d)$ over \mathbb{Q} , since it is fixed by the involution $\sigma\tau^{-1}$:

$$(f_w(Q_1) f_w(Q_2))^{\sigma\tau^{-1}} = f_w(Q_1)^{(1+\tau^{-1})\sigma\tau^{-1}} = f_w(Q_1)^{\sigma(1+\tau^{-1})} = f_w(Q_1)^{1+\tau^{-1}}.$$

In addition, $\omega = e^{2\pi i/3}$ is also fixed by $\sigma\tau^{-1}$, since $\omega^{\sigma\tau^{-1}} = \bar{\omega}^{\tau^{-1}} = \bar{\omega}^2 = \omega$.

More generally, a similar computation shows that each of the numbers $f_w(Q_1) f_w(Q_c)$ is fixed by the involution $\sigma\tau_c^{-1}$, so all the pairwise products of roots of $\tilde{W}_{-d}(x)$ have degree at most $h(-d)$. Thus, the irreducible factors of the resultant

$$R_d(x) = \text{Res}_y \left(y^{h(-d)} \tilde{W}_{-d}(x/y), \tilde{W}_{-d}(y) \right), \quad (11.2)$$

one of which is the minimal polynomial $\tilde{t}(x)$ of $f_w(Q_1)^3 f_w(Q_2)^3$ over \mathbb{Q} , have degree at most $h(-d)$, and $\tilde{t}(x)$ has the property that $\tilde{t}(x^3)$ also factors into the product of three irreducibles of degree $h(-d)$, as we show in the following theorem.

Theorem 11.1. If $3 \mid d$ and $d > 15$, the numbers $f_w(Q_c)f_w(Q_{2c})$ lie in Ω_f and form a complete system of conjugates over \mathbb{Q} (and over $K = \mathbb{Q}(\sqrt{-d})$) of

$$f_w(Q_1)f_w(Q_2) = (-1)^{(-d-1)/8} \zeta^{-v(3-(v^2+d)/8-(v^2+d)^2/32)} \mathfrak{f}(w/2)\mathfrak{f}_2(w),$$

as $\mathfrak{c} = (c, (v + \sqrt{-d})/2)$ varies over ideals in the $h(-d)$ ideal classes in \mathbf{R}_{-d} , with $c = \text{Norm}(\mathfrak{c})$, $(c, 6d) = 1$, and $16c \mid (v^2 + d)$.

Proof. The formula for $f_w(Q_1)f_w(Q_2)$ is immediate from (10.7) and (10.10). It is also clear that the numbers $f_w(Q_c)f_w(Q_{2c})$ are all conjugate to $f_w(Q_1)f_w(Q_2)$, by (10.15). We will show that these numbers are distinct, as the ideal \mathfrak{c} varies over distinct ideal classes in \mathbf{R}_{-d} . Assume that \mathfrak{c} is an ideal with $\text{Norm}(\mathfrak{c}) = c$, satisfying the conditions of the theorem, for which

$$f_w(Q_1)f_w(Q_2) = f_w(Q_c)f_w(Q_{2c}) = (f_w(Q_1)f_w(Q_2))^{\tau_c^{-1}}. \quad (11.3)$$

Raising (11.1) to the 4-th power gives

$$\begin{aligned} (f_w(Q_1)f_w(Q_2))^4 &= \frac{\omega^l}{2^4} \left(\frac{1}{4} \beta^{\tau^2} (\beta - 2) \right)^4 f_w(Q_2)^{12} \\ &= \frac{\omega^l}{2^4} (-\beta(\beta^2 + 4)) \left(\frac{4\alpha_1}{\beta^2} \right)^2 = \frac{-\omega^l(\beta^2 + 4)}{\beta^3} \alpha_1^2. \end{aligned}$$

Squaring this equation gives

$$(f_w(Q_1)f_w(Q_2))^8 = \frac{\omega^{2l}(\beta^2 + 4)^2}{\beta^6} \frac{-16\beta^4}{\beta^4 - 16} = \frac{-16\omega^{2l}(\beta^2 + 4)}{\beta^2(\beta^2 - 4)},$$

and (11.3) yields

$$\frac{\omega^{2l}(\beta^2 + 4)}{\beta^2(\beta^2 - 4)} = \frac{\omega^{2lc}(\beta_c^2 + 4)}{\beta_c^2(\beta_c^2 - 4)}. \quad (11.4)$$

Assume first that $\omega^l = 1$ or $\tau_c = (\Omega_f/K, \mathfrak{c})$ fixes ω . Then

$$\frac{(\beta^2 + 4)}{\beta^2(\beta^2 - 4)} = \frac{(\beta_c^2 + 4)}{\beta_c^2(\beta_c^2 - 4)}$$

implies that

$$(\beta^2 - \beta_c^2) ((\beta^2 + 4)\beta_c^2 + 4(\beta^2 - 4)) = 0.$$

Now $\beta^2 = \beta_c^2$ implies $\beta = \beta_c$ by Proposition 8.1. Thus, if $\beta \neq \beta_c$, then $\beta_c^2 = -4(\beta^2 - 4)/(\beta^2 + 4)$. However, comparing ideal factorizations, we have

$$(\beta_c^2) = (\beta^2)^{\tau_c^{-1}} = \wp_2^2 \wp_2'^4, \quad \left(\frac{-4(\beta^2 - 4)}{\beta^2 + 4} \right) = \frac{\wp_2^2 \wp_2'^2 \wp_2'^5 \wp_2'^2}{\wp_2^3 \wp_2'^2} = \wp_2^4 \wp_2'^2.$$

Since these two ideals are distinct, this shows that $\beta = \beta_c$ and therefore $\tau_c = 1$.

Now if τ_c satisfies (11.3), then so does τ_c^2 , which certainly fixes ω . The above argument shows that $\tau_c^2 = 1$, so any nontrivial automorphism satisfying (11.3) must have order 2. If there are two nontrivial automorphisms τ_{c_1} and τ_{c_2} satisfying (11.3), then so does the product, which fixes ω , so again we have $\tau_{c_1} \tau_{c_2} = 1$ and $\tau_{c_1} = \tau_{c_2}$. Hence, at most one nontrivial automorphism τ_c can satisfy (11.3), it can only have order 2, and it does not fix ω .

Assume that there is a nontrivial automorphism τ_c which satisfies (11.4), where $\omega^{lc} \neq \omega^l$. We apply the automorphism σ (complex conjugation) to both sides of (11.4), using the fact that $\beta_c^\sigma = \beta^{\tau_c^{-1}\sigma} = \beta^{\sigma\tau_c^{-1}}$:

$$\frac{\omega^l(\beta - 2)^2(\beta^2 + 4)}{\beta(\beta + 2)^2} = \frac{\omega^{lc}(\beta_c - 2)^2(\beta_c^2 + 4)}{\beta_c(\beta_c + 2)^2}. \quad (11.5)$$

Dividing (11.4) by (11.5) yields the equation

$$\frac{\omega^l(\beta + 2)}{\beta(\beta - 2)^3} = \frac{\omega^{lc}(\beta_c + 2)}{\beta_c(\beta_c - 2)^3}. \quad (11.6)$$

In order to find the common solutions of (11.4) and (11.6) we set

$$g_1(x, y) = (x^2 + 4)y^2(y^2 - 4) - \omega^{2l(c-1)}(y^2 + 4)x^2(x^2 - 4),$$

$$g_2(x, y) = (x + 2)y(y - 2)^3 - \omega^{l(c-1)}(y + 2)x(x - 2)^3,$$

where the equations $g_1(x, y) = 0, g_2(x, y) = 0$ have the common solution (β, β_c) . Taking the resultant of these equations gives

$$\text{Res}_y(g_1(x, y), g_2(x, y)) = 8(\omega^{l(c-1)} - 1)(x + 2)x^4(x - 2)^4 g_3(x),$$

where $g_3(x) \in \mathbb{Q}(\omega)[x]$ with

$$\text{Norm}_{\mathbb{Q}(\omega)/\mathbb{Q}}(g_3(x)) = (x^8 + 224x^4 + 256)(x + 2)^2(x^2 - 2x + 8)^4(x^4 - 8x^3 + 20x^2 - 16x + 64)^4.$$

Since β must be a root of this polynomial, it is clear that $d = 7$ or $d = 15$. (Compare with the polynomials $B_7(x)$ and $B_{15}(x)$ in the proof of Proposition 8.1, and note

that β is not a root of the 8-th degree factor, since otherwise $\beta^4 = -112 \pm 64\sqrt{3}$ and $\sqrt{-1} \in \Omega_f$.) Thus, no such automorphism τ_c exists when $d > 15$, and the proof is complete. \square

Corollary. If $3 \mid d$ and $d > 15$, the field L generated by $f_w(Q_1)f_w(Q_2)$ over \mathbb{Q} contains the field $\mathbb{Q}(\omega)$.

Proof. By the theorem, we have $[L : \mathbb{Q}] = h(-d)$, so $L = \text{Fix}(\sigma\tau^{-1})$. \square

If $t_d(x)$ is the minimal polynomial of $f_w(Q_1)f_w(Q_2)$ over \mathbb{Q} , then $t_d(x)$ factors into two irreducible polynomials of degree $h(-d)/2$ over $\mathbb{Q}(\omega)$, and has no real roots. Table 2 below lists the polynomial $t_d(x)$ and its discriminant for $39 \leq d \leq 495$ and $3 \mid d$. Note also that the polynomial $\tilde{t}(x^3)$ mentioned above is the product of $t_d(x)$ and the minimal polynomials over \mathbb{Q} of the numbers $\omega f_w(Q_1)f_w(Q_2)$ and $\omega^2 f_w(Q_1)f_w(Q_2)$, both of which have degree $h(-d)$.

12 Examples.

If we have the class equation $H_{-d}(x)$ we may compute the polynomial $b_d(x)$, the minimal polynomial of $\beta/2$, as the unique polynomial of degree $2h(-d)$ which divides

$$G_d(x) = x^{16h(-d)}(1 - x^4)^{h(-d)}H_{-d}\left(\frac{16(x^8 - 16x^4 + 16)^3}{x^{16}(1 - x^4)}\right) \quad (12.1)$$

and which is stabilized by the map $x \rightarrow (x + 1)/(x - 1)$. (The argument of $H_{-d}(x)$ in (12.1) is obtained from (6.2) by replacing β by $2x$.) The minimal polynomial $A_d(x)$ of $\zeta_8^j \alpha$ can be obtained in a similar way as a factor of degree $2h(-d)$ of the polynomial

$$\tilde{G}_d(x) = (x^8 + 16x^4)^{h(-d)}H_{-d}\left(\frac{(x^8 + 16x^4 + 16)^3}{x^8 + 16x^4}\right). \quad (12.2)$$

If the class equation $H_{-d}(x)$ is not available, then as in [36] or [32] the minimal polynomials $b_d(x)$ of $\beta/2$ and $A_d(x)$ of $\alpha_1 = \zeta_8^j \alpha$ can be computed using the infinite product representations of the Schläfli functions. For example, one may use representatives $\mathbf{c}_i = (c_i, (v_i + \sqrt{-d})/2)$ (with $16c_i \mid (v_i^2 + d)$) of the various ideal classes and compute the minimal polynomial of $\beta/2$ over K from Theorem 10.7:

Table 2: The minimal polynomial $t_d(x)$ of $f_w(Q_1)f_w(Q_2)$.

d	$t_d(x)$	$\text{disc}(t_d(x))$
39	$x^4 + 2x^3 + 2x^2 + x + 1$	$3^2 13$
63	$x^4 - x^3 + 3x^2 - x + 1$	$3^3 7$
87	$x^6 - x^5 + 4x^4 - 4x^3 + 5x^2 - 3x + 1$	$-3^3 29^2$
111	$x^8 + 2x^7 + 4x^6 + 8x^5 + 9x^4 + 7x^3 + 7x^2 + 4x + 1$	$3^4 11^2 37^3$
135	$x^6 + 3x^5 + 3x^4 + 2x^3 - 3x + 1$	$-3^7 5^2$
159	$x^{10} - x^9 + 6x^7 - x^6 - 6x^5 + 6x^4 + 8x^3 - 2x^2 - 3x + 1$	$-3^9 29^2 53^4$
183	$x^8 + x^7 + x^6 + 4x^5 + 12x^4 + 11x^3 + x^2 - x + 1$	$3^4 5^2 47^2 61^3$
207	$x^6 + 6x^5 + 16x^4 + 22x^3 + 16x^2 + 5x + 1$	$-3^3 23^2$
231	$x^{12} + 4x^{11} + 11x^{10} + 17x^9 + 20x^8 + 16x^7 + 2x^6 - 7x^5$ $+ 2x^4 + 13x^3 + 11x^2 + 2x + 1$	$3^6 7^4 11^6 17^4 31^2 71^2$
255	$x^{12} + 6x^{11} + 11x^{10} - 18x^8 - 6x^7 + 19x^6 + 12x^5 + 6x^4$ $+ 18x^3 + 14x^2 + 3x + 1$	$3^6 5^{14} 13^2 17^6$
279	$x^{12} - 4x^{11} + 3x^{10} + x^9 + 7x^8 - 9x^7 + x^6 - 6x^5 + 6x^4$ $+ 8x^3 + 16x^2 + 6x + 1$	$3^9 17^4 23^2 31^5 79^2$
303	$x^{10} - 5x^9 + 7x^8 + 2x^7 - 17x^6 + 23x^5 + 8x^4 - 42x^3$ $+ 32x^2 - 9x + 1$	$-3^{21} 101^4$
327	$x^{12} - 4x^{11} + 15x^{10} - 30x^9 + 48x^8 - 50x^7 + 44x^6 - 28x^5$ $+ 18x^4 - 6x^3 + 9x^2 + 7x + 1$	$3^{18} 5^2 109^5 191^2$
351	$x^{12} - 3x^{11} + 6x^{10} - 7x^9 + 15x^8 + 9x^7 + 23x^6 + 30x^5$ $+ 36x^4 + 19x^3 + 24x^2 + 9x + 1$	$3^{14} 13^5 101^2 151^2 251^2$
375	$x^{10} - 5x^8 + 5x^7 + 30x^6 + 24x^5 - 25x^4 - 30x^3 + 15x^2$ $+ 5x + 1$	$-3^5 5^{12} 7^4 41^2 239^2$
399	$x^{16} - 4x^{15} + 8x^{14} - 24x^{13} + 82x^{12} - 182x^{11} + 252x^{10}$ $- 223x^9 + 114x^8 + 5x^7 - 72x^6 + 61x^5 + 7x^4 - 45x^3$ $+ 35x^2 - 7x + 1$	$3^{14} 7^{12} 11^2 19^8 89^2$ $\cdot 199^2 227^2$
423	$x^{10} - 7x^9 + 21x^8 - 30x^7 + 22x^6 - 8x^5 + 19x^4 - 28x^3$ $+ 30x^2 - 2x + 1$	$-3^5 5^2 13^2 31^2 41^4 47^4$
447	$x^{14} - 3x^{13} + 9x^{12} - 28x^{11} + 44x^{10} - 21x^9 - 8x^8 - 29x^7$ $+ 97x^6 - 63x^5 + 11x^4 - 16x^3 + 33x^2 - 3x + 1$	$-3^{23} 5^{14} 13^2 17^4 149^6$
471	$x^{16} + 8x^{14} + 4x^{13} - 24x^{11} - 30x^{10} - 17x^9 + 81x^8 + 189x^7$ $+ 199x^6 + 162x^5 + 86x^4 + 49x^3 + 19x^2 - 7x + 1$	$3^{20} 11^{12} 17^4 157^7$ $\cdot 271^2 311^2$
495	$x^{16} - 2x^{15} + 13x^{14} - 27x^{13} + 62x^{12} - 59x^{11} + 39x^{10}$ $+ 29x^9 - 35x^8 - 20x^7 + 129x^6 - 169x^5 + 146x^4 - 57x^3$ $+ 16x^2 + 8x + 1$	$3^{12} 5^{14} 11^8 23^4 29^2$ $\cdot 79^2 103^2 367^2$

$$m(x) = \prod_{i=1}^{h(-d)} \left(x - i^{-v_i c_i} \frac{\mathfrak{f}_1(w_i/2)^2}{\mathfrak{f}(w_i/2)^2} \right), \quad w_i = \frac{v_i + \sqrt{-d}}{2c_i}. \quad (12.3)$$

It is necessary to take $v_i \equiv v_k \pmod{8}$ for $i \neq k$ in this product, in order for the ideal (β_{c_i}) to be $\wp_2 \wp_2'^2$. Then $b_d(x) = m(x)\bar{m}(x)$. The polynomial $A_d(x)$ can be found in a similar way using (10.20), or can be determined as the factor of the polynomial

$$\tilde{A}(x) = \text{Res}_y(b_d(y), (1 - y^4)x^4 - 16y^4),$$

which has $\alpha_1 = \zeta_8^j \alpha$ from (10.20) as a root.

In Tables 3 and 4 we list the polynomials $b_d(x)$ and $A_d(x)$ for $15 \leq d \leq 103$. In each case, the polynomial $A_d(x^2) = A_1(x)A_1(-x)$ factors, verifying Theorem 8.10 that $\zeta_8^j \alpha$ is a square in the relevant class field. In Table 5 we list the minimal polynomial $q_d(x)$ of the unit $4\alpha_1/\beta^2$. This polynomial may be computed as the unique irreducible factor of degree $h(-d)$ of the resultant

$$\text{Res}_y(A_d(xy), s_d(y)), \quad s_d(y) = \text{Res}_x(b_d(x), y - x^2). \quad (12.4)$$

By Theorems 10.2 and 10.3, the minimal polynomial $W_{-d}(x)$ of the unit $f_w(Q_1)$ in (10.10) divides $q_d(x^6)$ if $(d, 3) = 1$, and the minimal polynomial $\tilde{W}_{-d}(x)$ of $f_w(Q_1)^3$ divides $q_d(x^2)$ when $3 \mid d$.

Remarks. 1. The polynomial $A_d(x)$ has the property that $x^{2h(-d)}A_d(4/x) = 2^{2h(-d)}A_d(x)$. The polynomials $b_d(x)$ and $A_d(x)$ are of course normal polynomials, so they provide interesting examples of the elementary theory discussed in [15].

2. Since it is a relatively simple matter to compute the polynomial $b_d(x)$ using (12.3), the class equation $H_{-d}(x)$ can be computed using the resultant

$$\text{Res}_y(b_d(y), y^{16}(1 - y^4)x - 16(y^8 - 16y^4 + 16)^3).$$

Computing this resultant mod p allows one to compute $H_{-d}(x) \pmod{p}$ directly from the polynomial $b_d(x)$. We used the polynomials $b_d(x)$ in Table 3 to compute the points Q_K in Section 8.

3. In the case $d = 55$ roots π and ξ of $b_{55}(x) = 0$ satisfying $\pi^4 + \xi^4 = 1$ can be computed using the fact that $b_{55}(x)$ is invariant under $x \rightarrow (x+1)/(x-1)$:

$$\pi = \frac{-3 + \sqrt{5} + 3\sqrt{-11} - \sqrt{-55}}{4} + \left(\frac{11 + 4\sqrt{-11} + \sqrt{-55}}{44} \right) \sqrt{-138 + 62\sqrt{5}},$$

Table 3: The minimal polynomial $b_d(x)$ of $\beta/2$.

d	$b_d(x)$
15	$x^4 - 4x^3 + 5x^2 - 2x + 4$
23	$x^6 + x^5 + 9x^4 - 13x^3 + 18x^2 - 16x + 8$
31	$x^6 + 7x^5 + 11x^4 - 15x^3 + 16x^2 - 20x + 8$
39	$x^8 - 6x^7 + 42x^6 - 60x^5 + 53x^4 - 54x^3 + 24x^2 + 16$
47	$x^{10} - 15x^9 + 74x^8 - 90x^7 + 93x^6 - 187x^5 + 160x^4 - 156x^3 + 168x^2 - 48x + 32$
55	$x^8 + 6x^7 + 78x^6 - 84x^5 + 53x^4 - 66x^3 - 12x^2 + 24x + 16$
63	$x^8 + 20x^7 + 110x^6 - 100x^5 + 49x^4 - 80x^3 - 40x^2 + 40x + 16$
71	$x^{14} - 11x^{13} + 195x^{12} - 127x^{11} + 473x^{10} - 593x^9 + 489x^8 - 1285x^7 + 1858x^6 - 2880x^5 + 3320x^4 - 2656x^3 + 1792x^2 - 576x + 128$
79	$x^{10} - 31x^9 + 290x^8 - 186x^7 + 5x^6 - 251x^5 - 56x^4 - 60x^3 + 256x^2 + 32x + 32$
87	$x^{12} + 16x^{11} + 395x^{10} + 398x^9 - 357x^8 - 316x^7 - 155x^6 - 1058x^5 + 1332x^4 - 704x^3 + 800x^2 - 352x + 64$
95	$x^{16} + 42x^{15} + 508x^{14} - 260x^{13} - 78x^{12} + 3160x^{11} - 2072x^{10} + 2204x^9 + 53x^8 - 9378x^7 + 12004x^6 - 16216x^5 + 16208x^4 - 10144x^3 + 6016x^2 - 2048x + 256$
103	$x^{10} - 21x^9 + 732x^8 - 290x^7 - 191x^6 - 369x^5 - 502x^4 + 40x^3 + 456x^2 + 144x + 32$

Table 4: The minimal polynomial $A_d(x)$ of $\alpha_1 = \zeta_8^j \alpha$.

d	$A_d(x)$
15	$x^4 + x^3 - 3x^2 + 4x + 16$
23	$x^6 - 2x^5 + 5x^4 + 3x^3 + 20x^2 - 32x + 64$
31	$x^6 - x^5 + 10x^4 - 17x^3 + 40x^2 - 16x + 64$
39	$x^8 - 8x^7 + 26x^6 - 33x^5 + 29x^4 - 132x^3 + 416x^2 - 512x + 256$
47	$x^{10} - 7x^9 + 11x^8 + x^7 + 66x^6 - 241x^5 + 264x^4 + 16x^3 + 704x^2 - 1792x + 1024$
55	$x^8 + 5x^7 + 17x^6 - 15x^5 - 67x^4 - 60x^3 + 272x^2 + 320x + 256$
63	$x^8 - 9x^7 + 58x^6 - 240x^5 + 609x^4 - 960x^3 + 928x^2 - 576x + 256$
71	$x^{14} + 3x^{13} - 2x^{12} - 18x^{11} + 291x^{10} - 602x^9 + 1069x^8 - 573x^7 + 4276x^6 - 9632x^5 + 18624x^4 - 4608x^3 - 2048x^2 + 12288x + 16384$
79	$x^{10} + 16x^9 + 96x^8 + 271x^7 + 406x^6 + 495x^5 + 1624x^4 + 4336x^3 + 6144x^2 + 4096x + 1024$
87	$x^{12} - 5x^{11} + 34x^{10} + 20x^9 + 337x^8 - 35x^7 + 989x^6 - 140x^5 + 5392x^4 + 1280x^3 + 8704x^2 - 5120x + 4096$
95	$x^{16} + 8x^{15} + 82x^{14} + 369x^{13} + 1357x^{12} + 3847x^{11} + 9905x^{10} + 20781x^9 + 42909x^8 + 83124x^7 + 158480x^6 + 246208x^5 + 347392x^4 + 377856x^3 + 335872x^2 + 131072x + 65536$
103	$x^{10} - 3x^9 - 21x^8 + 120x^7 + 1121x^6 + 3267x^5 + 4484x^4 + 1920x^3 - 1344x^2 - 768x + 1024$

Table 5: The minimal polynomial $q_d(x)$ of $4\alpha_1/\beta^2$.

d	$q_d(x)$
15	$x^2 - 3x + 1$
23	$x^3 + 2x^2 + 5x - 1$
31	$x^3 - x^2 + 10x - 1$
39	$x^4 + 4x^3 + 2x^2 - 17x + 1$
47	$x^5 + 9x^4 + 27x^3 + 33x^2 + 26x - 1$
55	$x^4 - 7x^3 + 29x^2 - 43x + 1$
63	$x^4 - x^3 + 18x^2 - 64x + 1$
71	$x^7 - 9x^6 + 34x^5 - 42x^4 + 91x^3 + 50x^2 + 93x - 1$
79	$x^5 - 8x^4 + 63x^2 + 134x - 1$
87	$x^6 + 7x^5 + 46x^4 + 44x^3 + 217x^2 - 191x + 1$
95	$x^8 - 4x^7 + 58x^6 - 95x^5 + 401x^4 - 445x^3 - 243x^2 - 263x + 1$
103	$x^5 + 17x^4 + 119x^3 + 340x^2 + 361x - 1$

and

$$\xi = \frac{-3 + \sqrt{5} - 3\sqrt{-11} + \sqrt{-55}}{4} + \left(\frac{-11 + 4\sqrt{-11} + \sqrt{-55}}{44} \right) \sqrt{-138 + 62\sqrt{5}}.$$

4. For $d = 63$ a solution (π_3, ξ_3) of $\pi_3^4 + \xi_3^4 = 1$ is

$$\begin{aligned} \pi_3 &= \frac{-5 - \sqrt{-3} + \sqrt{-7} + \sqrt{21}}{2} + \frac{1 + \sqrt{-7}}{2} \sqrt{-9 + 2\sqrt{21}}, \\ \xi_3 &= \frac{-5 + \sqrt{-3} - \sqrt{-7} + \sqrt{21}}{2} - \frac{1 - \sqrt{-7}}{2} \sqrt{-9 + 2\sqrt{21}}. \end{aligned}$$

These numbers lie in the ring class field Ω_3 over $K = \mathbb{Q}(\sqrt{-7})$ with conductor $f = 3$. We note that π_3 is divisible by $\pi_1 = (1 - \sqrt{-7})/2$ (not by $\xi_1 = (1 + \sqrt{-7})/2$!) and that π_3/π_1 is a unit in Ω_3 :

$$\frac{\pi_3}{\pi_1} = \frac{-6 + 3\sqrt{-3} - 2\sqrt{-7} + \sqrt{21}}{4} + \frac{-3 + \sqrt{-7}}{4} \sqrt{-9 + 2\sqrt{21}}.$$

We illustrate the above discussion by working out the details of the case $d = 159$, an example that was also considered in [36]. We use the following ideal basis quotients:

$$\begin{aligned} w_1 &= \frac{1 + \sqrt{-159}}{2}, & w_2 &= \frac{1 + \sqrt{-159}}{2 \cdot 5}, & w_3 &= \frac{9 + \sqrt{-159}}{2 \cdot 5}, & w_4 &= \frac{17 + \sqrt{-159}}{2 \cdot 7}, \\ w_5 &= \frac{25 + \sqrt{-159}}{2 \cdot 7}, & w_6 &= \frac{25 + \sqrt{-159}}{2 \cdot 49}, & w_7 &= \frac{33 + \sqrt{-159}}{2 \cdot 13}, & w_8 &= \frac{41 + \sqrt{-159}}{2 \cdot 23}, \\ w_9 &= \frac{89 + \sqrt{-159}}{2 \cdot 101}, & w_{10} &= \frac{-87 + \sqrt{-159}}{2 \cdot 23}; \end{aligned}$$

the formula (12.3) then yields the polynomial

$$\begin{aligned} b_{159}(x) &= x^{20} + 128x^{19} + 4661x^{18} - 12230x^{17} + 40058x^{16} - 14824x^{15} - 30958x^{14} \\ &+ 105436x^{13} - 148835x^{12} + 147288x^{11} - 30615x^{10} - 86502x^9 + 152488x^8 - 309376x^7 \\ &+ 268880x^6 - 255776x^5 + 217664x^4 - 85120x^3 + 50432x^2 - 12800x + 1024. \end{aligned}$$

From this we use the resultant $\tilde{A}(x)$ to compute

$$\begin{aligned} A_{159}(x) &= x^{20} + 5x^{19} + 153x^{18} + 89x^{17} + 1401x^{16} - 947x^{15} + 5229x^{14} + 3580x^{13} \\ &+ 82934x^{12} - 7773x^{11} + 88893x^{10} - 31092x^9 + 1326944x^8 + 229120x^7 + 1338624x^6 \\ &- 969728x^5 + 5738496x^4 + 1458176x^3 + 10027008x^2 + 1310720x + 1048576. \end{aligned}$$

Now we compute the polynomial $q_{159}(x)$ using (12.4):

$$\begin{aligned} q_{159}(x) &= x^{10} + x^9 + 141x^8 + 757x^7 + 3397x^6 + 10593x^5 + 18265x^4 \\ &+ 14032x^3 + 2454x^2 - 2501x + 1, \end{aligned}$$

and then $q_{159}(x^2) = \tilde{W}_{-159}(x)\tilde{W}_{-159}(-x)$, where

$$\tilde{W}_{-159}(x) = x^{10} - x^9 + x^8 + 7x^7 + 63x^6 + 121x^5 + 219x^4 + 196x^3 + 146x^2 + 47x - 1,$$

as in [36, p. 1661]. Next we compute the resultant $R_{159}(x)$ in (11.2):

$$R_{159}(x) = p_1(x)p_2(x)^2p_3(x)^2p_4(x)^2p_5(x)^2p_6(x)^2,$$

where all but one of the degrees of the $p_i(x)$ are 10, and $\deg(p_2(x)) = 5$, with

$$p_2(x) = x^5 - 8x^4 + 43x^3 - 101x^2 + 91x + 1.$$

The polynomials

$$p_3(x) = x^{10} + 17x^9 + 105x^8 + 312x^7 + 584x^6 + 642x^5 + 444x^4 + 245x^3 + 181x^2 - 21x + 1$$

and

$$p_4(x) = x^{10} + 4x^9 + 43x^8 + 86x^7 - 2x^6 - 2218x^5 + 5285x^4 - 5199x^3 + 2195x^2 + 27x + 1$$

have the property that $p_3(x^3)$ and $p_4(x^3)$ split into 10-th degree factors. Of these six factors, the one with the smallest discriminant is the minimal polynomial of $v_1 = \omega f_w(Q_1)f_w(Q_2)$, which is a factor of $p_3(x^3)$:

$$t_1(x) = x^{10} - 4x^9 + 12x^8 - 21x^7 + 29x^6 - 27x^5 + 18x^4 - 7x^3 + x^2 + 1,$$

and $\text{disc}(t_1(x)) = -3^5 11^2 53^4$. This discriminant is smaller than the discriminant $3^8 19^2 53^5$ of the most efficient polynomial

$$x^{10} - x^9 + 2x^8 + 7x^7 + x^6 - 15x^5 - 5x^4 + 8x^3 + 5x^2 - 5x + 1 \quad (12.5)$$

given in [36, p. 1661], even though $t_1(x)$ has slightly larger coefficients. (The polynomial $t_{159}(x)$ in Table 2 has smaller discriminant *and* smaller height than (12.5).) We note that the polynomial in (12.5) is a factor of $\text{Res}_y(A_1(xy), A_1(y))$, where $A_1(x)$ is the minimal polynomial of $\sqrt{\alpha_1}$, and so its roots are quotients of conjugates of $\sqrt{\alpha_1}$. This resultant is divisible by eight other 10th degree polynomials of relatively small height and discriminant.

Of course $t_1(x)$ is irreducible over K , so that we have $\Sigma = K(v_1)$. Note that $t_1(x)$ has no real roots, and factors over $\mathbb{Q}(\sqrt{-3})$ as the product of the polynomial

$$x^5 + (2\omega - 1)x^4 + (-3\omega + 1)x^3 + (4\omega + 1)x^2 + (-2\omega - 1)x + 1$$

and its conjugate (with discriminant $53^2\omega$). This implies that an integral basis for $L = \mathbb{Q}(v_1)$ over \mathbb{Q} is the set $\{v_1^i, \omega v_1^i, 0 \leq i \leq 4\}$. In addition, $p_2(x^3)$ has the convenient quintic factor

$$t_2(x) = x^5 - 2x^4 + x^3 - 2x^2 + 4x + 1,$$

where $\text{disc}(t_2(x)) = 3^4 53^2$. Since $t_2(x)$ is irreducible over $K(\sqrt{-3})$ we have that $\Sigma = K(v_2, \sqrt{-3})$, where v_2 is a root of $t_2(x)$. If v_2 is the real root of $t_2(x)$, then the real subfield of Σ is $\Sigma^+ = \mathbb{Q}(v_2, \sqrt{53})$.

For the sake of comparison we note that the minimal polynomial of the unit γ (see Theorem 8.6) is

$$u_{159}(x) = x^{10} + 59x^9 + 918x^8 - 4696x^7 + 8545x^6 - 6567x^5 + 2029x^4 - 79x^3 - 63x^2 + 5x + 1,$$

and $u_{159}(x^2) = r(x)r(-x)$, where

$$r(x) = x^{10} + x^9 + 30x^8 - 48x^7 - 39x^6 + 55x^5 + 29x^4 - 9x^3 - 7x^2 - 3x - 1.$$

We have $\text{disc}(r(x)) = 3^{12}31^247^253^579^2$, and so the root $\sqrt{\gamma}$ (taking the positive sign in Proposition 10.8) of $r(x)$ is not as efficient a generator of Ω_f/K as is the root v_1 of $t_1(x)$.

In conclusion, we see that the arithmetic of the Weber singular moduli yields natural generators of the ring class fields Ω_f over $K = \mathbb{Q}(\sqrt{-d})$, with small height and discriminant, even when 3 divides the discriminant $-d$. A short table of the minimal polynomials of such generators for discriminants divisible by 3 is given in Table 2.

13 References.

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